Pre-lecture brain teaser

Assume L is any regular language. Let's define a new language:

Definition
$\text{Flip}(L) = \{\bar{w} \mid w \in L, x \in \Sigma^*\}$

CS/ECE-374: Lecture 6 - Regular Languages - Closure Properties

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February 11, 2021

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Yes Next problem.

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Also yes.

Closure propeties

Definition
(Informal) A set A is **closed** under an operation **op** if applying **op** to any elements of A results in an element that also belongs to A.

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(Informal) A set A is **closed** under an operation **op** if applying **op** to any elements of A results in an element that also belongs to A.

Examples:

- *Integers:* closed under $+$, $-$, $*$, but not division.
- *Positive integers:* closed under $+$ but not under $-$
- *Regular languages:* closed under union, intersection, Kleene star, complement, difference, homomorphism, inverse homomorphism, reverse, $\ldots$

Closure properties of Regular Languages

How do we prove that regular languages are closed under some new operation?

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Three broad approaches

- Use existing closure properties

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Three broad approaches

- Use existing closure properties
  - L_1, L_2, L_3, L_4 regular implies $(L_1 - L_2) \cap (\overline{L_3} \cup L_4)^*$ is regular
- Transform regular expressions
- Transform DFAs to NFAs — versatile technique and shows the power of nondeterminism

Homomorphism closure

Let's look back at the pre-lecture teaser. Define a function

$$h(x) = \begin{cases} 1 & x = 0 \\ 0 & x = 1 \end{cases}$$

This is known as a homomorphism - A cipher that is a one-to-one mapping to one character set to another.

How do we prove $h(L)$ is regular if L is regular?

Homomorphism closure

Proof Idea:

1. Suppose R is a regular expression for L.
2. We define $Flip(L) = L^F$ as a regular expression based off the regular expression for L (using a finite number of concatenations, unions and Kleene Star)
3. Thus L^F is regular because it has a regular expression.

Thus we reduce the argument to $L(h(R)) = h(L(R))$

Homomorphism closure

Let's define the regular expression inductively by transforming the operations in R. We see that:

- **Base Case:** Zero operators in R means that $R =: a \in \Sigma$, ε, $\emptyset$. In any case we define $R^F = h(R)$
- Otherwise R has three potential types of operators to transform. Splitting R at an operator we see:
  - $h(R_1 R_2) = h(R_1) \cdot h(R_2)$
  - $h(R_1 \cup R_2) = h(R_1) \cup h(R_2)$
  - $h(R^*) = (h(R))^*$

Hence, since we can define L^F via a regular language, L^F is regular.

Regular Languages

Regular languages have three different characterizations

- Inductive definition via base cases and closure under union, concatenation and Kleene star
- Languages accepted by DFAs
- Languages accepted by NFAs

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Regular language closed under many operations:

- union, concatenation, Kleene star via inductive definition or NFAs
- complement, union, intersection via DFAs
- homomorphism, inverse homomorphism, reverse, ...

Different representations allow for flexibility in proofs.

Closure problem - Reverse

Example: REVERSE

Given string w, w^R is reverse of w.

For a language L define $L^R = \{w^R \mid w \in L\}$ as reverse of L.

Theorem
L^R is regular if L is regular.

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Infinitely many regular languages!

Proof technique:

- take some finite representation of L such as regular expression r
- Describe an algorithm A that takes r as input and outputs a regular expression r' such that $L(r') = (L(r))^R$.
- Come up with A and prove its correctness.

REVERSE via regular expressions

Suppose r is a regular expression for L. How do we create a regular expression r' for L^R?

REVERSE via regular expressions

Suppose r is a regular expression for L. How do we create a regular expression r' for L^R? Inductively based on recursive definition of r.

- $r = \emptyset$ or $r = a$ for some $a \in \Sigma$
- $r = r_1 + r_2$
- $r = r_1 \cdot r_2$
- $r = (r_1)^*$

REVERSE via regular expressions

- $r = \emptyset$ or $r = a$ for some $a \in \Sigma$
  $r' =$
- $r = r_1 + r_2$.
  If r_1', r_2' are reg expressions for $(L(r_1))^R, (L(r_2))^R$ then
  $r' =$
- $r = r_1 \cdot r_2$.
  If r_1', r_2' are reg expressions for $(L(r_1))^R, (L(r_2))^R$ then
  $r' =$
- $r = (r_1)^*$.
  If r_1' is reg expressions for $(L(r_1))^R$ then
  $r' =$

$r = (0 + 10)^*(001 + 01)1$ then $r' =$

REVERSE via machine transformation

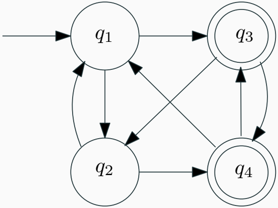
Given DFA $M = (Q, \Sigma, \delta, s, A)$ want NFA N such that $L(N) = (L(M))^R$.

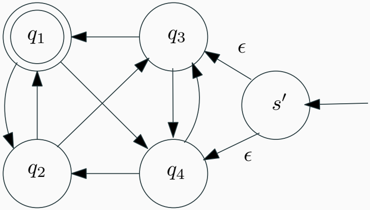
N should accept w^R **iff** M accepts w

M accepts w iff $\delta^*_M(s, w) \in A$

Idea:

REVERSE via machine transformation





Caveat: Reversing transitions may create an NFA.

REVERSE via machine transformation

Proof (DFA to NFA): Let $M = (\Sigma, Q, s, A, \delta)$ be an arbitrary DFA that accepts L. We construct an NFA $M^R = (\Sigma, Q^R, s^R, A^R, \delta^R)$ with ε-transitions that accepts L^R, intuitively by reversing every transition in M, and swapping the roles of the start state and the accepting states. Because M does not have a unique accepting state, we need to introduce a special start state s^R, with ε-transitions to each accepting state in M. These are the only ε-transitions in M^R.

$$\begin{aligned} Q^R &= Q \cup \{s^R\} \\ A^R &= \{s\} \\ \delta^R(s^R, \varepsilon) &= A \\ \delta^R(s^R, a) &= \varnothing && \text{for all } a \in \Sigma \\ \delta^R(q, \varepsilon) &= \varnothing && \text{for all } q \in Q \\ \delta^R(q, a) &= \left\{ p \;\middle|\; q \in \delta(p, a) \right\} && \text{for all } q \in Q \text{ and } a \in \Sigma \end{aligned}$$

Routine inductive definition-chasing now implies that the reversal of any sequence $q_0 \to q_1 \to \cdots \to q_\ell$ of transitions in M is a valid sequence $q_\ell \to q_{\ell-1} \to \cdots \to q_0$ of transitions in M^R. Because the transitions retain their labels (but reverse directions), it follows that M accepts any string w if and only if M^R accepts w^R.

We conclude that the NFA M^R accepts L^R, so L^R must be regular. $\square$

REVERSE via machine transformation

Formal proof: two directions

- $w \in L(M)$ implies $w^R \in L(N)$. Sketch. Let $\delta^*_M(s, w) = q$ where $q \in A$. On input w^R N non-deterministically transitions from its start state s' to q on an ϵ transition, and traces the reverse of the walk of M on w^R and hence reaches s which is an accepting state of N. Thus N accepts w^R

- $u \in L(N)$ implies $u^R \in L(M)$. Sketch. If $u \in N$ it implies that s' transitioned to some $q \in A$ on ϵ transition and

Closure Problem - Cycle

A more complicated example: CYCLE

$CYCLE(L) = \{yx \mid x, y \in \Sigma^*, xy \in L\}$

Theorem
$CYCLE(L)$ is regular if L is regular.

Example: $L = \{abc, 374a\}$

$CYCLE(L) =$

A more complicated example: CYCLE

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Theorem
$CYCLE(L)$ is regular if L is regular.

Given DFA M for L create NFA N that accepts $CYCLE(L)$.

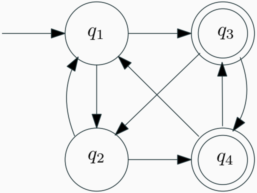
- N is a finite state machine, cannot know split of w into xy and yet has to simulate M on x and y.
- Exploit fact that M is itself a finite state machine. N only needs to “know” the state $\delta_M^*(s, x)$ and there are only finite number of states in M

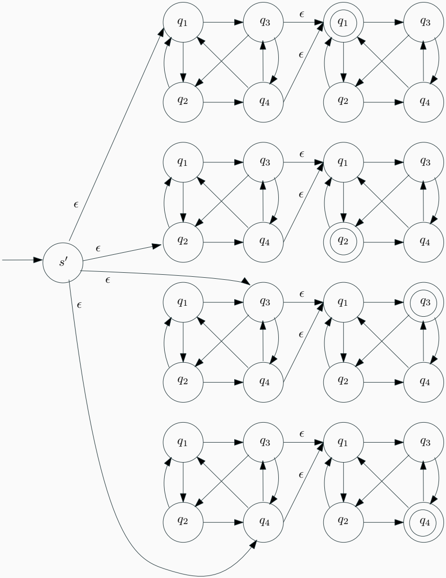
Construction for CYCLE

Let $w = xy$ and $w' = yx$.

- *N guesses* state $q = \delta_M^*(s, x)$ and simulates M on w' with start state q.
- *N guesses* when y ends (at that point M must be in an accept state) and transitions to a copy of M to simulate M on remaining part of w' (which is x)
- N accepts w' if after second copy of M on x it ends up in the guessed state q

Construction for CYCLE





Proving correctness

Exercise: Write down formal description of N in tuple notation starting with $M = (Q, \Sigma, \delta, s, A)$.

Need to argue that $L(N) = CYCLE(L(M))$

- If $w = xy$ accepted by M then argue that yx is accepted by N
- If N accepts w' then argue that $w' = yx$ such that xy accepted by M.

Closure Problem - Prefix

Example: PREFIX

Let L be a language over Σ.

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Example: PREFIX

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$Z = X \cap Y$

Create new DFA $M' = (Q, \Sigma, \delta, s, Z)$

Example: PREFIX

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If L is regular then PREFIX(L) *is regular.*

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Create new DFA $M' = (Q, \Sigma, \delta, s, Z)$

Claim: $L(M') = \mathrm{PREFIX}(L)$.

Exercise: SUFFIX

Let L be a language over Σ.

Definition
$\text{SUFFIX}(L) = \{w \mid xw \in L, x \in \Sigma^*\}$

Prove the following:

Theorem
If L is regular then SUFFIX(L) is regular.

Exercise: SUFFIX

Let L be a language over Σ.

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$\text{SUFFIX}(L) = \{w \mid xw \in L, x \in \Sigma^*\}$

Prove the following:

Theorem
If L is regular then SUFFIX(L) is regular.

Same idea as PREFIX(L)

$X = \{q \in Q \mid s \text{ can reach } q \text{ in } M\}$

$Y = \{q \in Q \mid q \text{ can reach some state in } A\}$

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With one major **difference**:

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$L_1 = \bar{L_3} \cap 0^*1^*$ hence if L_3 is regular then L_1 is regular, a contradiction