NP and NP Completeness

Lecture 23

Tuesday, November 29, 2022

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23.1
NP-Completeness: Cook-Levin Theorem

23.1.1
Completeness

NP: Non-deterministic polynomial

Definition 23.1.

A decision problem is in **NP**, *if it has a polynomial time certifier, for all the all the YES instances.*

Definition 23.2.

A decision problem is in **co-NP**, *if it has a polynomial time certifier, for all the all the NO instances.*

Example 23.3.

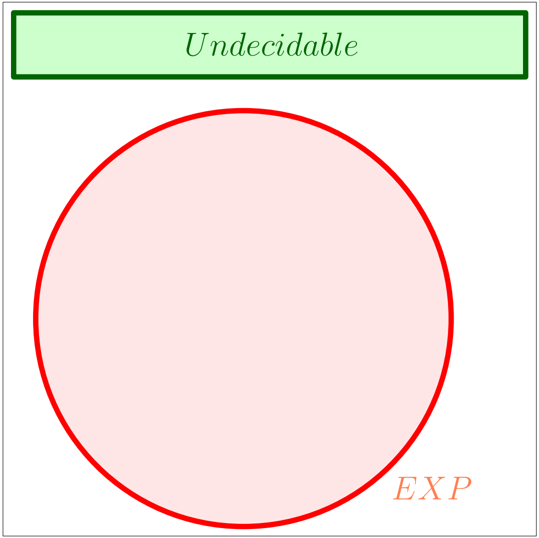
1. **3SAT** is in **NP**.
2. But **Not3SAT** is in **co-NP**.

In the beginning...

In the beginning...

$\boxed{\textit{Undecidable}}$

In the beginning...



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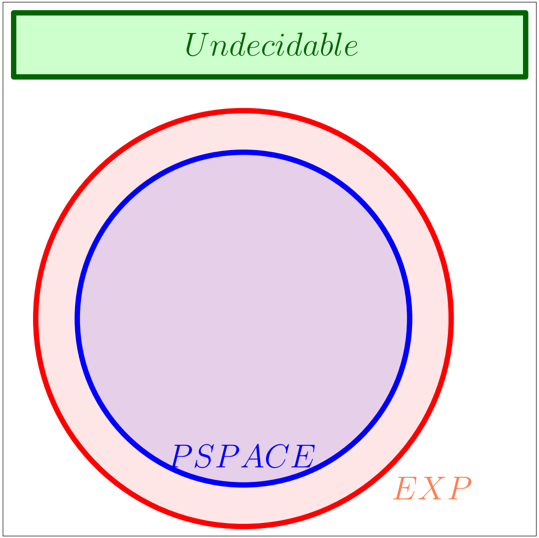

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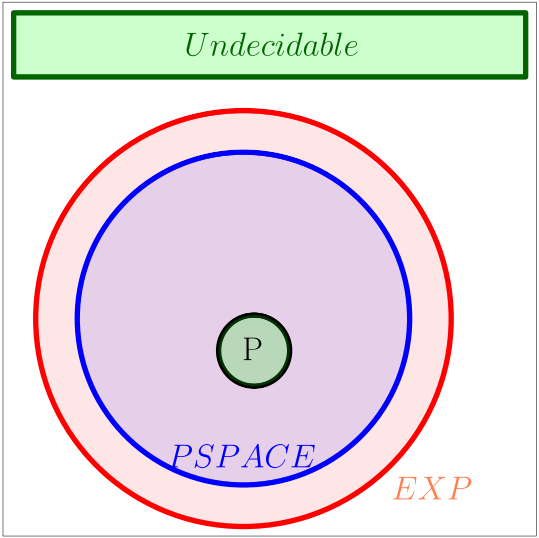

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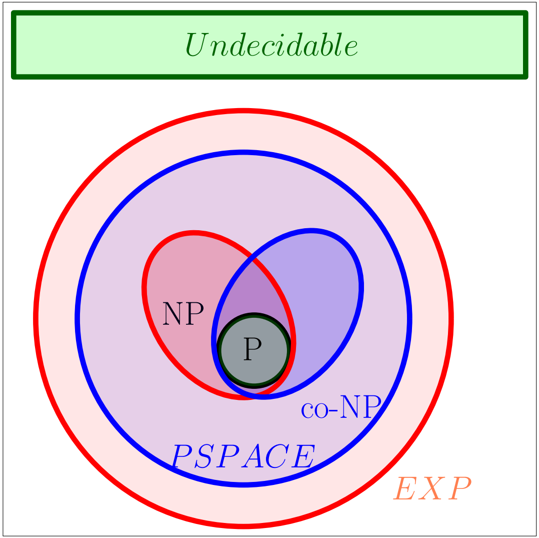

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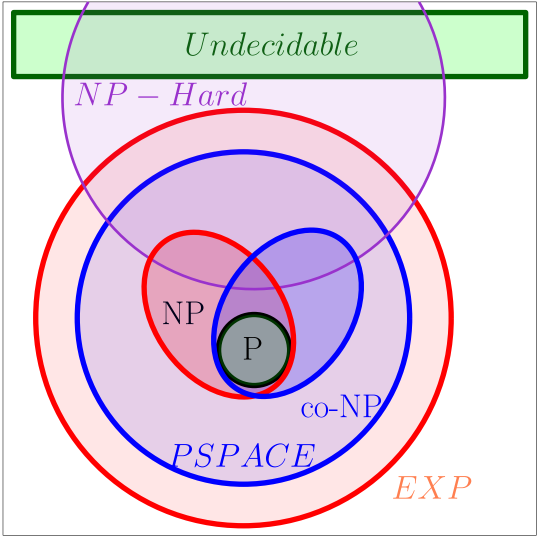

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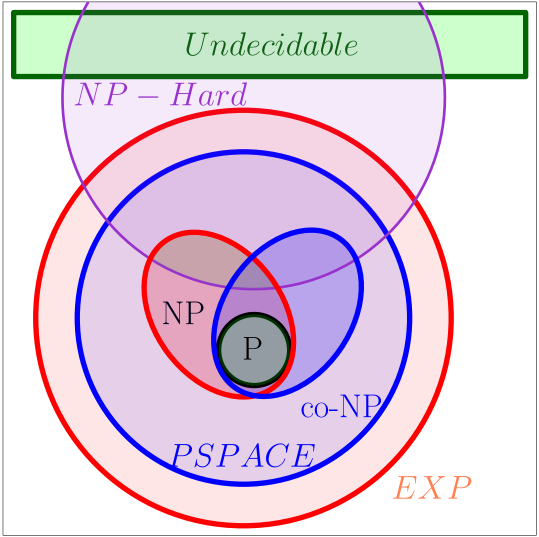

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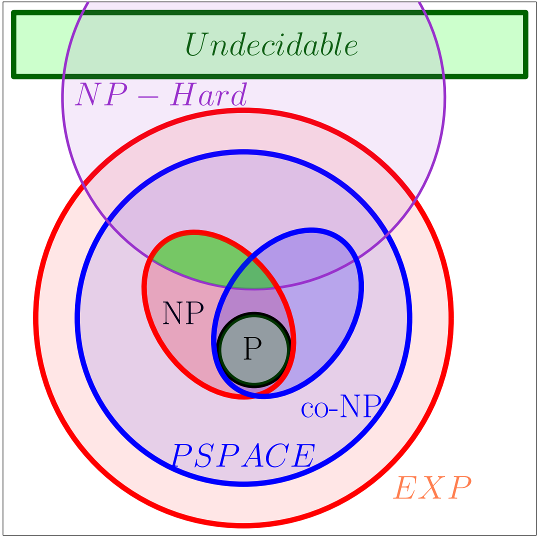

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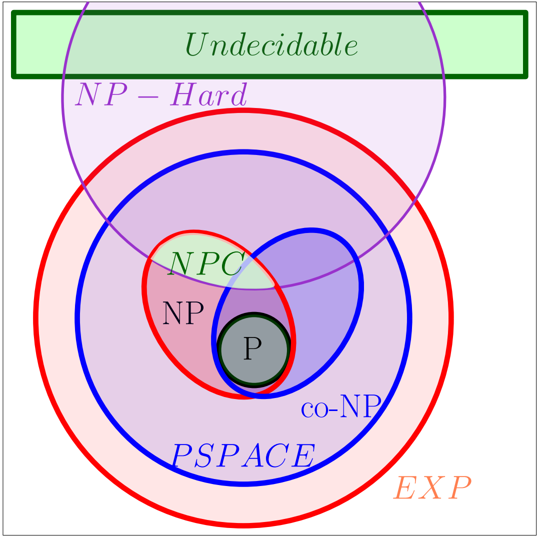

"Hardest" Problems

Question

What is the hardest problem in **NP**? How do we define it?

Towards a definition

1. Hardest problem must be in **NP**.
2. Hardest problem must be at least as "difficult" as every other problem in **NP**.

NP-Complete Problems

Definition 23.4.

A problem $\mathbf{X}$ is said to be **NP-Complete** if

1. $\mathbf{X} \in \mathbf{NP}$, and
2. (Hardness) For any $\mathbf{Y} \in \mathbf{NP}$, $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$.

Solving **NP-Complete** Problems

Proposition 23.5.

Suppose $\mathbf{X}$ *is* **NP-Complete**. *Then* $\mathbf{X}$ *can be solved in polynomial time* $\iff$ $\mathbf{P} = \mathbf{NP}$.

Proof.

$\Rightarrow$ Suppose $\mathbf{X}$ can be solved in polynomial time

0.1 Let $\mathbf{Y} \in \mathbf{NP}$. We know $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$.

0.2 We showed that if $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$ and $\mathbf{X}$ can be solved in polynomial time, then $\mathbf{Y}$ can be solved in polynomial time.

0.3 Thus, every problem $\mathbf{Y} \in \mathbf{NP}$ is such that $\mathbf{Y} \in \mathbf{P}$.

0.4 $\implies \mathbf{NP} \subseteq \mathbf{P}$.

0.5 Since $\mathbf{P} \subseteq \mathbf{NP}$, we have $\mathbf{P} = \mathbf{NP}$.

$\Leftarrow$ Since $\mathbf{P} = \mathbf{NP}$, and $\mathbf{X} \in \mathbf{NP}$, we have a polynomial time algorithm for $\mathbf{X}$. $\square$

NP-Hard Problems

Definition 23.6.

A problem $\mathbf{X}$ is said to be **NP-Hard** if

1. (Hardness) For any $\mathbf{Y} \in \mathbf{NP}$, we have that $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$.

An **NP-Hard** problem need not be in **NP**!

Example: Halting problem is **NP-Hard** (why?) but not **NP-Complete**.

Consequences of proving **NP-Complete**ness

If **X** is **NP-Complete**

1. Since we believe $\mathbf{P} \neq \mathbf{NP}$,
2. and solving **X** implies $\mathbf{P} = \mathbf{NP}$.

X is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for **X**.
(This is proof by mob opinion — take with a grain of salt.)

Consequences of proving **NP-Completeness**

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23.1.2
SAT is NP-Complete

NP-Complete Problems

Question

Are there any problems that are **NP-Complete**?

Answer

Yes! Many, many problems are **NP-Complete**.

Cook-Levin Theorem

Theorem 23.7 (Cook-Levin).

SAT *is* **NP-Complete**.

Need to show

1. **SAT** is in **NP**.
2. every **NP** problem **X** reduces in polynomial time to **SAT**.

Might see proof later...

Steve Cook won the Turing award for his theorem.

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23.1.3
Other NP Complete Problems

Proving that a problem **X** is **NP-Complete**

To prove **X** is **NP-Complete**, show

1. Show that **X** is in **NP**.
2. Give a polynomial-time reduction from a known **NP-Complete** problem such as **SAT** to **X**

SAT $\leq_P$ **X** implies that every **NP** problem **Y** $\leq_P$ **X**. Why?
Transitivity of reductions:

Y $\leq_P$ **SAT** and **SAT** $\leq_P$ **X** and hence **Y** $\leq_P$ **X**.

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3-SAT is NP-Complete

- **3-SAT** is in **NP**
- **SAT** $\leq_{\mathbf{P}}$ **3-SAT** as we saw

NP-Completeness via Reductions

1. **SAT** is **NP-Complete** due to Cook-Levin theorem
2. **SAT** $\leq_P$ **3-SAT**
3. **3-SAT** $\leq_P$ **Independent Set**
4. **Independent Set** $\leq_P$ **Vertex Cover**
5. **Independent Set** $\leq_P$ **Clique**
6. **3-SAT** $\leq_P$ **3-Color**
7. **3-SAT** $\leq_P$ **Hamiltonian Cycle**

Hundreds and thousands of different problems from many areas of science and engineering have been shown to be **NP-Complete**.

A surprisingly frequent phenomenon!

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23.2
Reducing **3-SAT** to Independent Set

Independent Set

Problem: Independent Set

Instance: A graph G, integer $\mathbf{k}$.
Question: Is there an independent set in G of size $\mathbf{k}$?

Lemma 23.1.
Independent set *is in* **NP**.

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3SAT $\leq_P$ Independent Set

The reduction 3SAT $\leq_P$ Independent Set

Input: Given a $3\mathrm{CNF}$ formula φ

Goal: Construct a graph $\mathbf{G}_\varphi$ and number $\mathbf{k}$ such that $\mathbf{G}_\varphi$ has an independent set of size $\mathbf{k}$ if and only if φ is satisfiable.

$\mathbf{G}_\varphi$ should be constructable in time polynomial in size of φ

Importance of reduction: Although 3SAT is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

Notice: We handle only $3\mathrm{CNF}$ formulas – reduction would not work for other kinds of boolean formulas.

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Interpreting 3SAT

There are two ways to think about **3SAT**

1. Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
2. Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of **3SAT** to construct the reduction.

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The Reduction

1. $\mathbf{G}_\varphi$ will have one vertex for each literal in a clause
2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
4. Take **k** to be the number of clauses

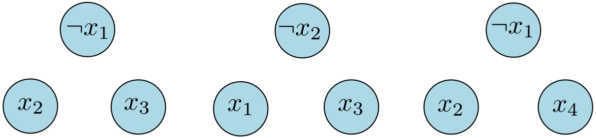

Figure: Graph for $\varphi = (\neg \mathbf{x_1} \vee \mathbf{x_2} \vee \mathbf{x_3}) \wedge (\mathbf{x_1} \vee \neg \mathbf{x_2} \vee \mathbf{x_3}) \wedge (\neg \mathbf{x_1} \vee \mathbf{x_2} \vee \mathbf{x_4})$

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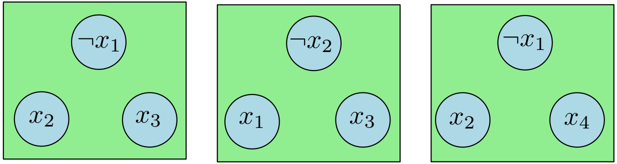

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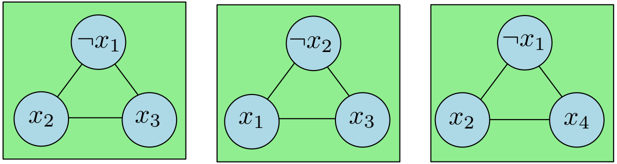

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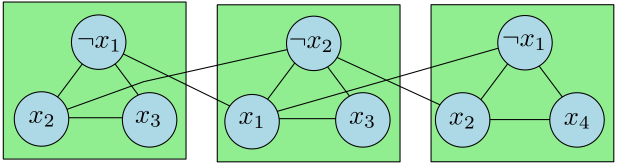

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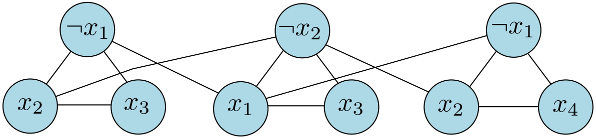

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Correctness

Proposition 23.2.

φ is satisfiable iff $\mathbf{G}_\varphi$ has an independent set of size $\mathbf{k}$ ($=$ number of clauses in φ).

Proof.

$\Rightarrow$ Let $\mathbf{a}$ be the truth assignment satisfying φ

- Pick one of the vertices, corresponding to true literals under $\mathbf{a}$, from each triangle. This is an independent set of the appropriate size. Why?

□

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Proof.

$\Leftarrow$ Let $\mathbf{S}$ be an independent set of size $\mathbf{k}$

1. $\mathbf{S}$ must contain exactly one vertex from each clause
2. $\mathbf{S}$ cannot contain vertices labeled by conflicting literals
3. Thus, it is possible to obtain a truth assignment that makes in the literals in $\mathbf{S}$ true; such an assignment satisfies one literal in every clause □

Summary

Theorem 23.3.

Independent set *is* **NP-Complete** *(i.e.,* **NPC***).*

23.3
NP-Completeness of Hamiltonian Cycle

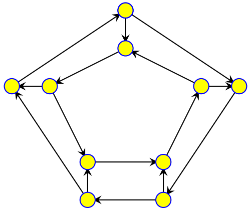
23.3.1
Reduction from 3SAT to Hamiltonian Cycle: Basic idea

Directed Hamiltonian Cycle

Input Given a directed graph $\mathbf{G = (V, E)}$ with $\mathbf{n}$ vertices

Goal Does $\mathbf{G}$ have a Hamiltonian cycle?

- A Hamiltonian cycle is a cycle in the graph that visits every vertex in $\mathbf{G}$ exactly once



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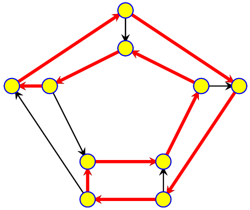
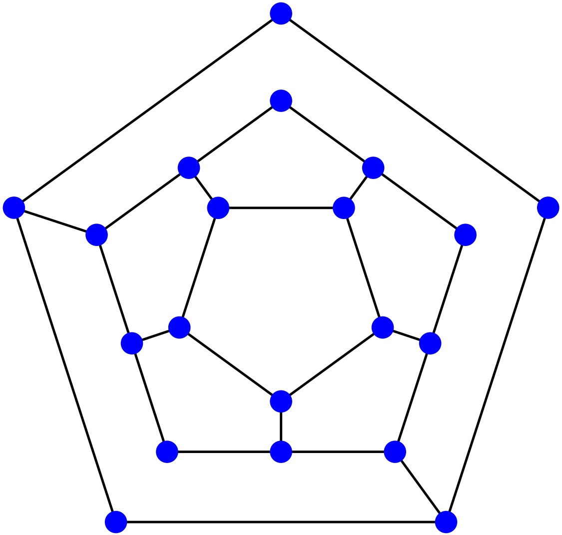


Is the following graph Hamiltonian?



(A) Yes.

(B) No.

Directed Hamiltonian Cycle is **NP-Complete**

- Directed Hamiltonian Cycle is in **NP**: exercise
- Hardness: We will show **3SAT** $\leq_P$ **Directed Hamiltonian Cycle** .

Reduction construction

From 3SAT to Hamiltonian cycle in directed graph

1. To show reduction, we next describe an algorithm:
   - Input: **3SAT** formula φ
   - Output: A graph $\mathbf{G}_\varphi$.
   - Running time is polynomial.
   - Requirement: φ is satisfiable $\iff$ $\mathbf{G}_\varphi$ is Hamiltonian.
2. Given **3SAT** formula φ create a graph G_φ such that
   - $\mathbf{G}_\varphi$ has a Hamiltonian cycle if and only if φ is satisfiable
   - $\mathbf{G}_\varphi$ should be constructible from φ by a polynomial time algorithm $\mathcal{A}$
3. Notation: φ has $\mathbf{n}$ variables $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ and $\mathbf{m}$ clauses $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_m$.

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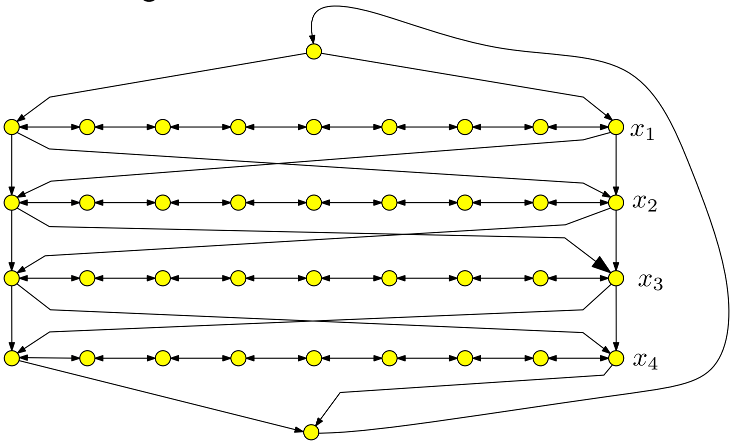
Encoding assignments

Converting φ to a graph

Given a formula with **n** variables, we need a graph with $\mathbf{2^n}$ different Hamiltonian paths, that can encode their assignments.



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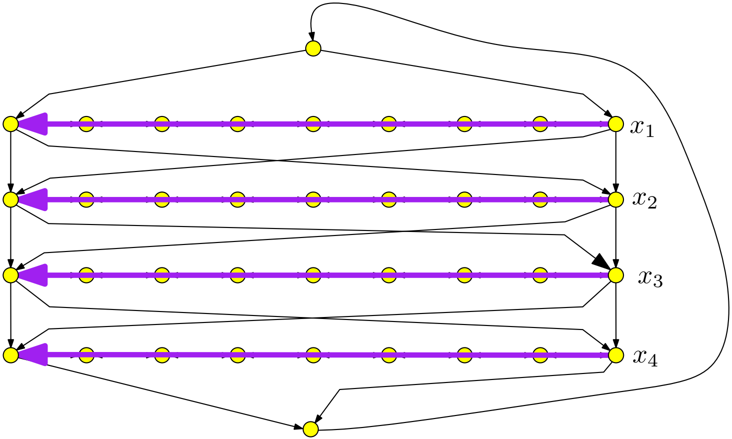
Encoding assignments

Converting φ to a graph

Given a formula with $\mathbf{n}$ variables, we need a graph with $\mathbf{2^n}$ different Hamiltonian paths, that can encode their assignments.



2

5

$$\mathbf{x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0}$$

Encoding assignments

Converting φ to a graph

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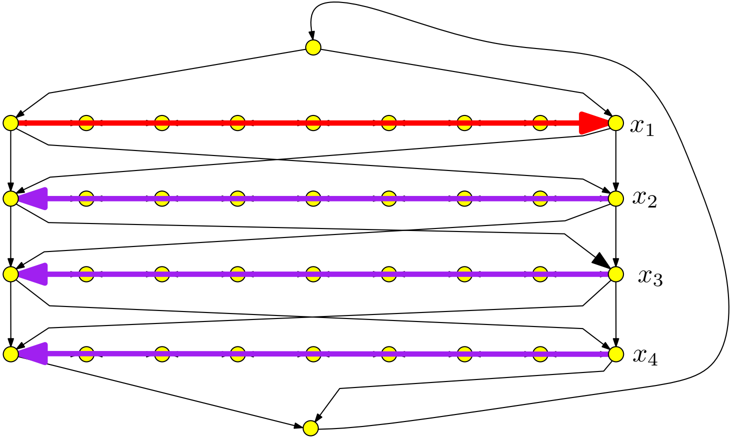

3

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$$\mathbf{x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 0}$$

Encoding assignments

Converting φ to a graph

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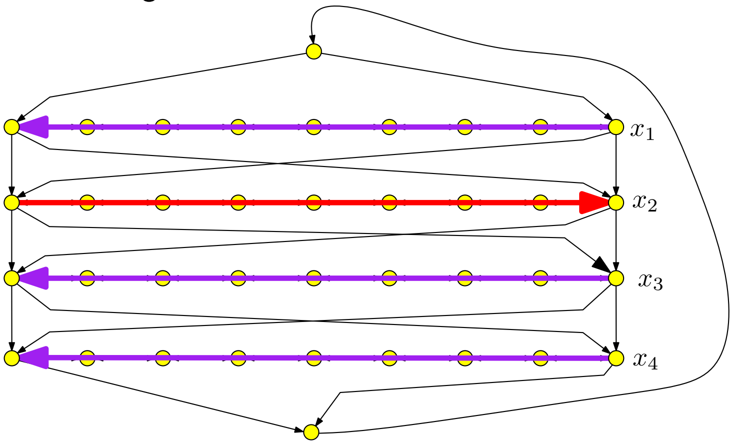

4

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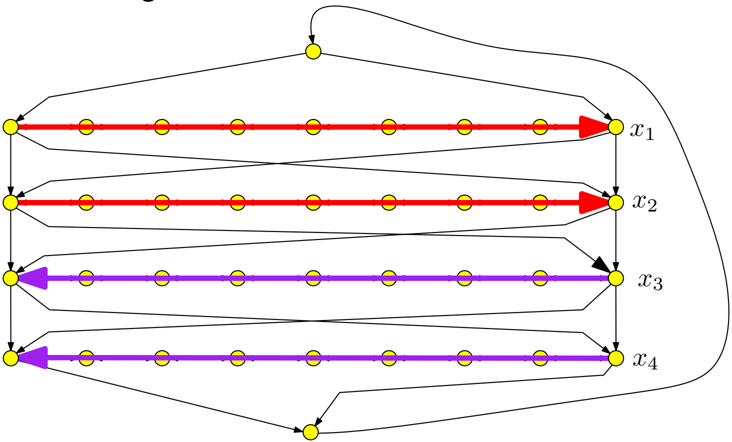

5

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$$\mathbf{x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0}$$

Encoding assignments

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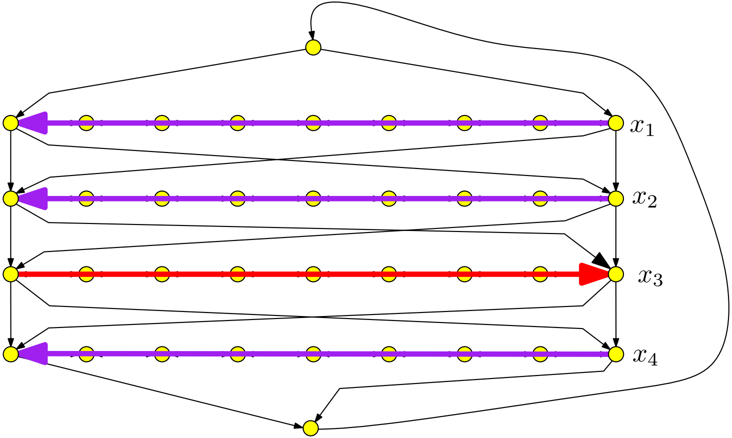

6

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$$\mathbf{x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 0}$$

Encoding assignments

Converting φ to a graph

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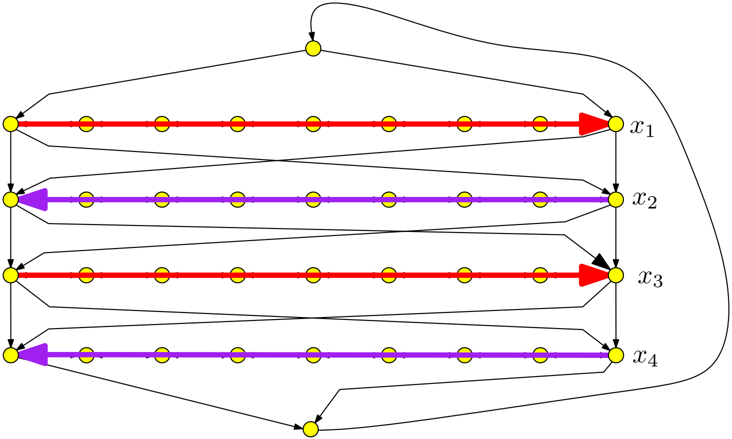

7

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$$\mathbf{x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0}$$

Encoding assignments

Converting φ to a graph

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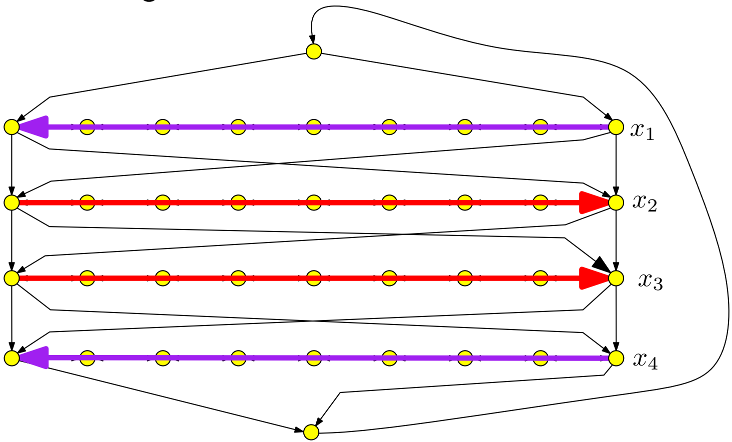

8

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$$\mathbf{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0}$$

Encoding assignments

Converting φ to a graph

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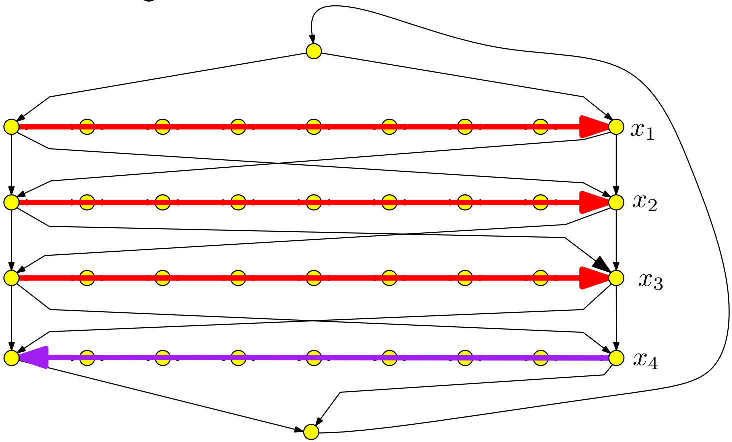

9

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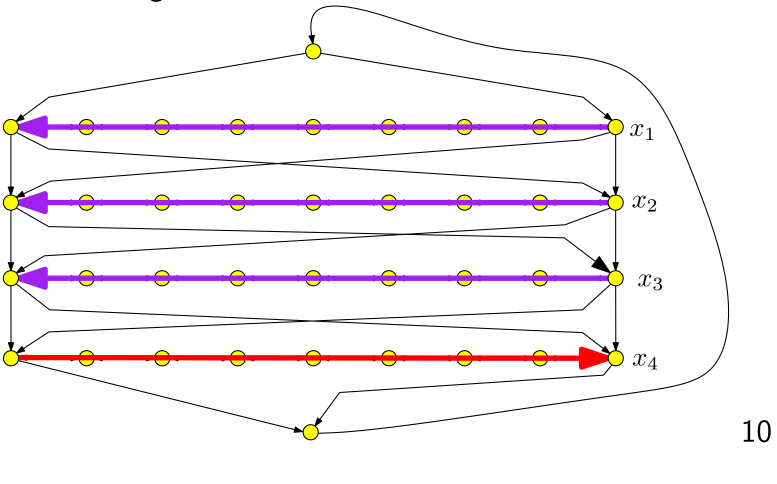
$$\mathbf{x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 01}$$

Encoding assignments

Converting φ to a graph

Given a formula with **n** variables, we need a graph with $\mathbf{2^n}$ different Hamiltonian paths, that can encode their assignments.



$$\mathbf{x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1}$$

Encoding assignments

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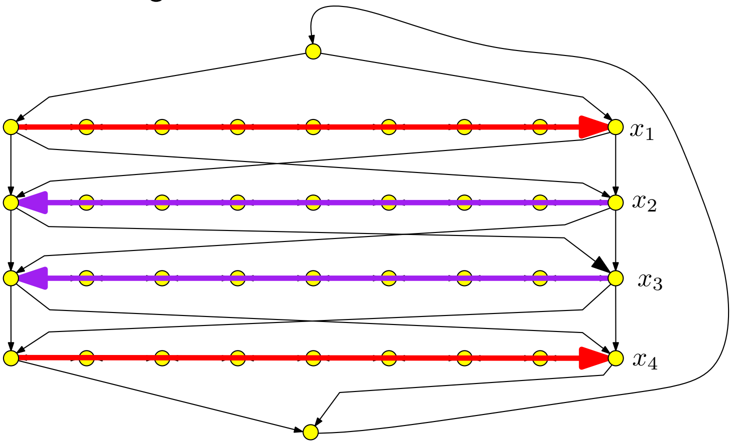

11

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$$\mathbf{x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1}$$

Encoding assignments

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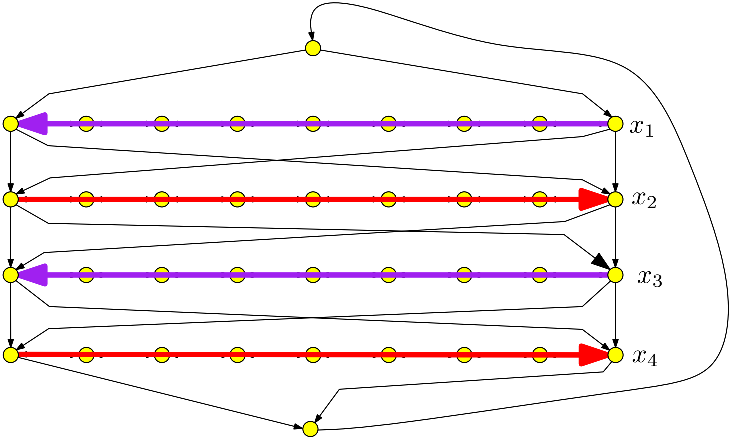


12

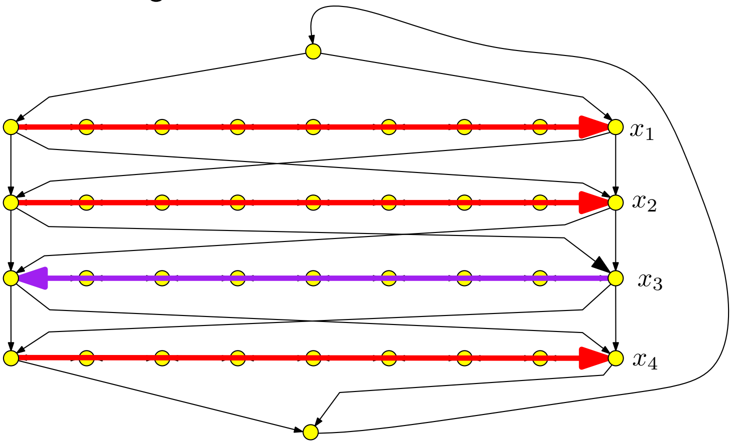
5

$$\mathbf{x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1}$$

Encoding assignments

Converting φ to a graph

Given a formula with $\mathbf{n}$ variables, we need a graph with $\mathbf{2^n}$ different Hamiltonian paths, that can encode their assignments.



13

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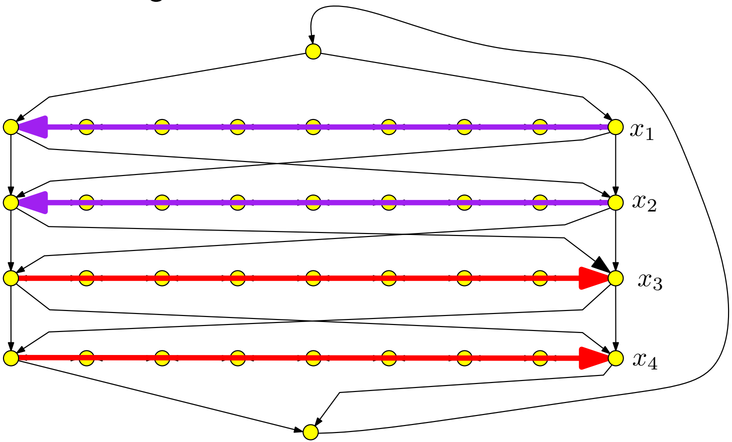
$$\mathbf{x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1}$$

Encoding assignments

Converting φ to a graph

Given a formula with $\mathbf{n}$ variables, we need a graph with $\mathbf{2^n}$ different Hamiltonian paths, that can encode their assignments.



14

5

$$\mathbf{x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1}$$

Encoding assignments

Converting φ to a graph

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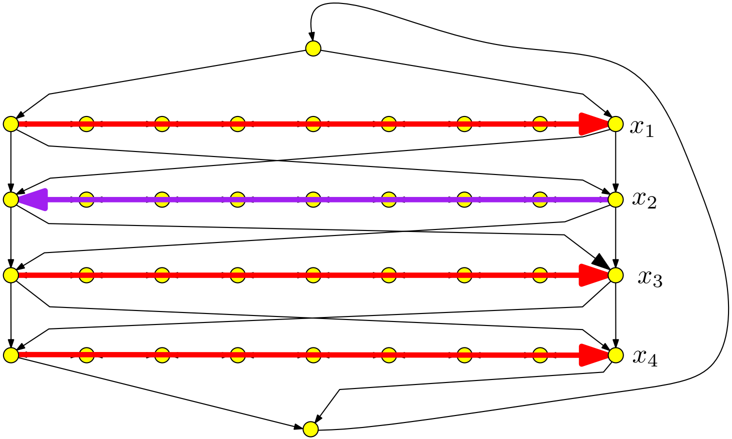

15

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$$\mathbf{x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1}$$

Encoding assignments

Converting φ to a graph

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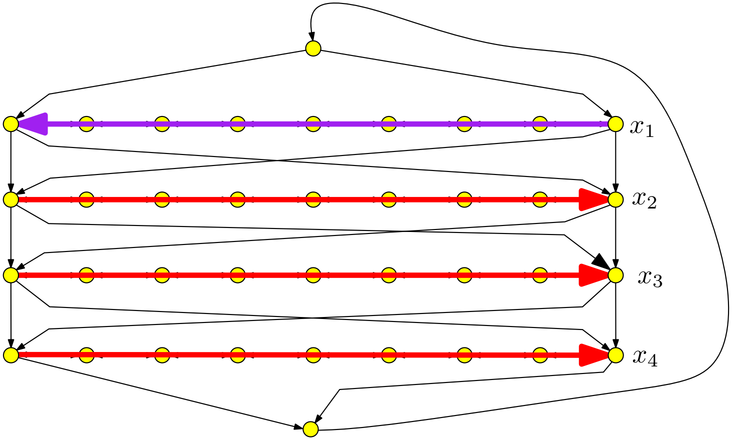

16

5

$$\mathbf{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 1}$$

Encoding assignments

Converting φ to a graph

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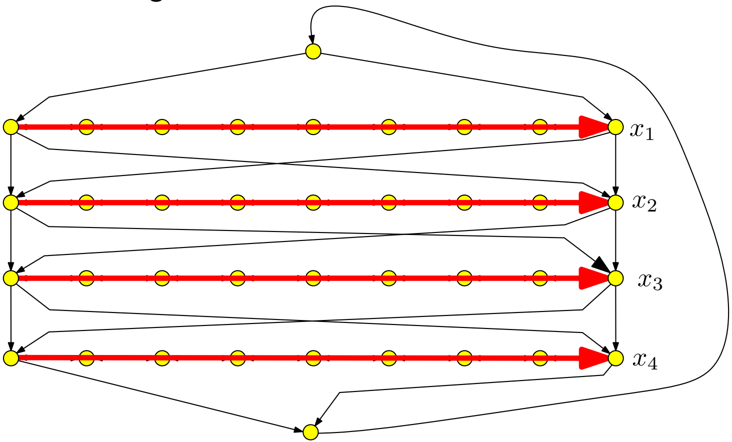

17

5

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Encoding assignments

Converting φ to a graph

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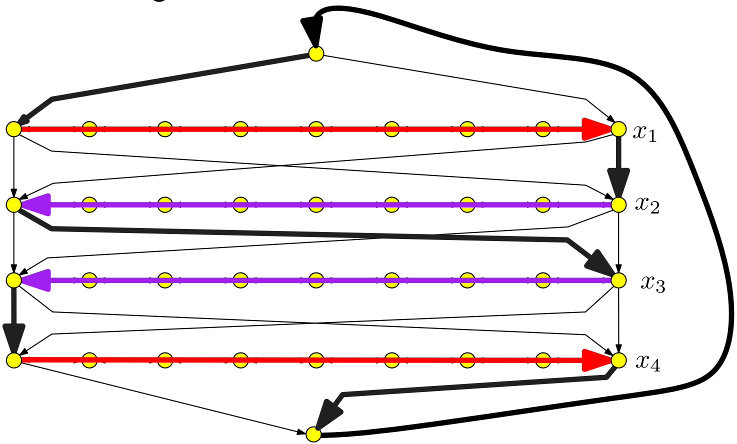

18

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23.3.2

The reduction: Encoding the formula constraints

3SAT $\leq_P$ Directed Hamiltonian Cycle

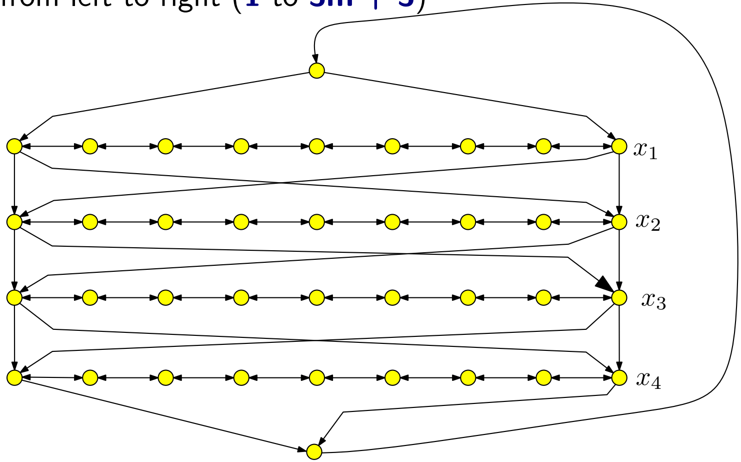
Input: φ formula.
Output: Graph $\mathbf{G}_\varphi$.

Saw: How to encode assignments...
Now need to encode constraints of φ.

The reduction algorithm: Phase I

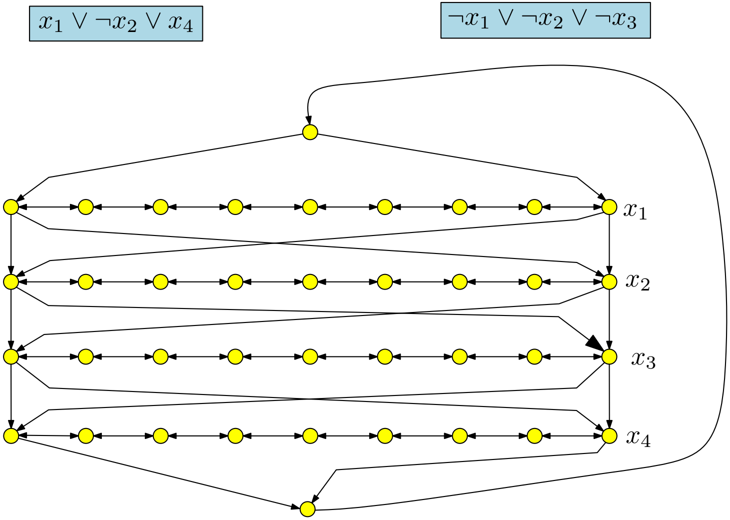
Converting φ to a graph

- Traverse path $\mathbf{i}$ from left to right iff $\mathbf{x_i}$ is set to true
- Each path has $\mathbf{3(m+1)}$ nodes where $\mathbf{m}$ is number of clauses in φ; nodes numbered from left to right ($\mathbf{1}$ to $\mathbf{3m+3}$)



The Reduction algorithm: Phase II

- Add vertex $\mathbf{c_j}$ for clause $\mathbf{C_j}$. $\mathbf{c_j}$ has edge from vertex $\mathbf{3j}$ and to vertex $\mathbf{3j+1}$ on path $\mathbf{i}$ if $\mathbf{x_i}$ appears in clause $\mathbf{C_j}$, and has edge from vertex $\mathbf{3j+1}$ and to vertex $\mathbf{3j}$ if $\mathbf{\neg x_i}$ appears in $\mathbf{C_j}$.



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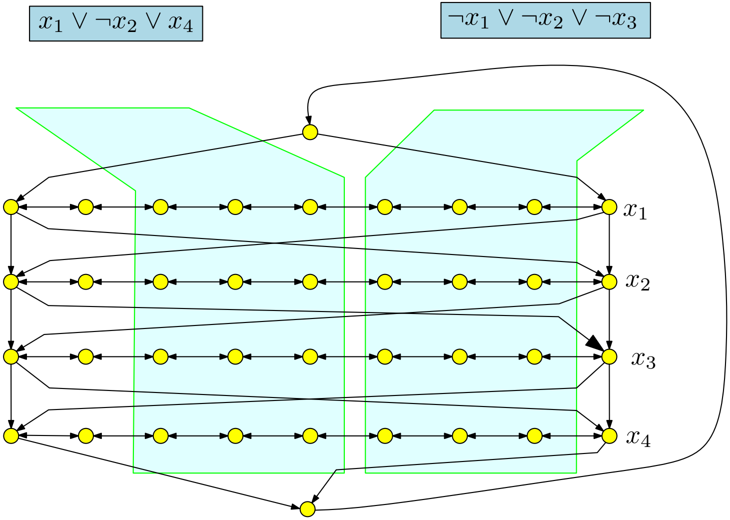

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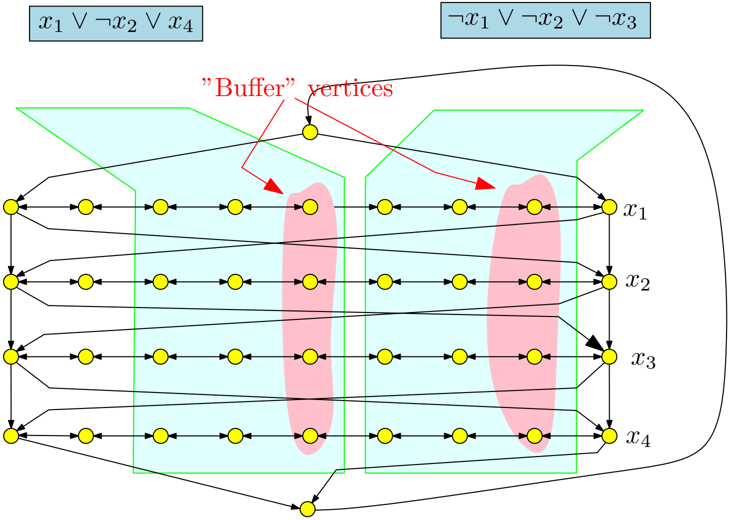

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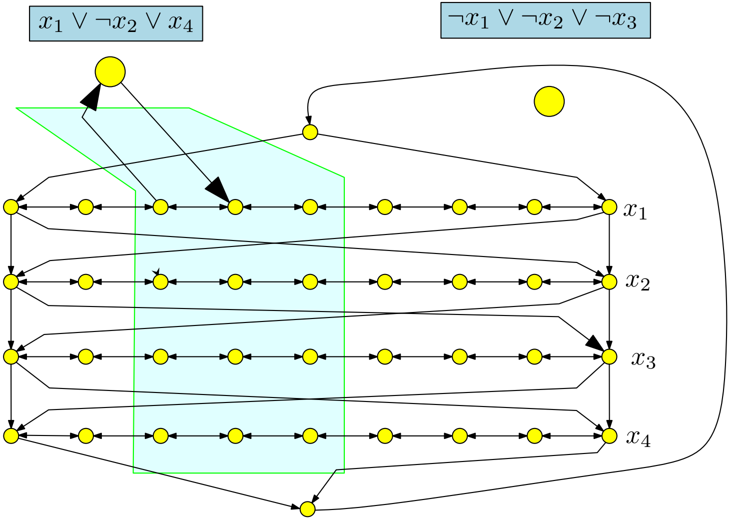

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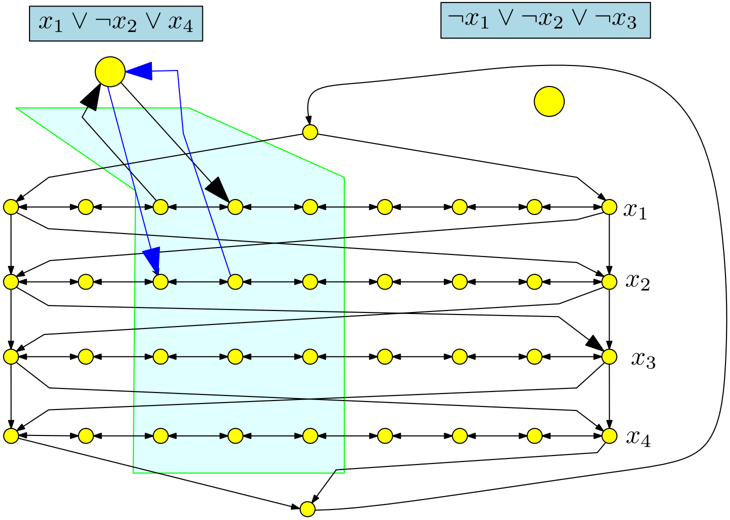

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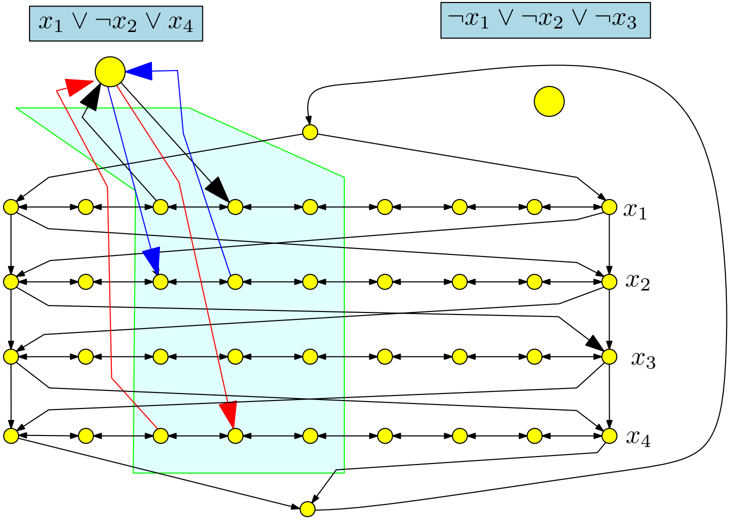

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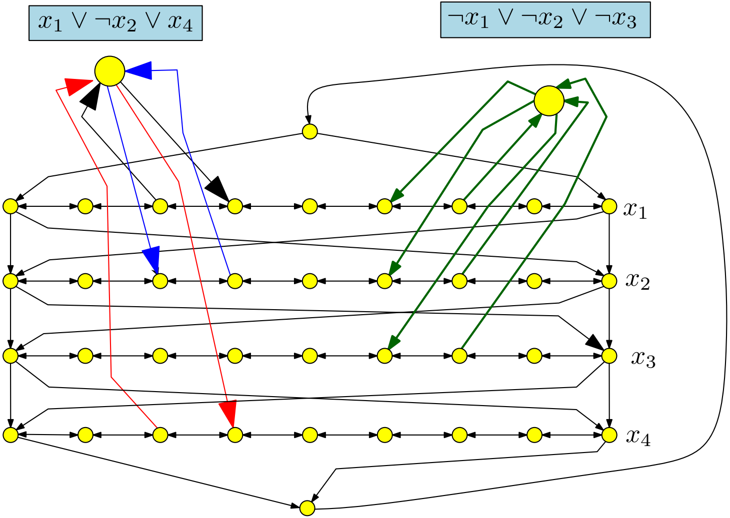
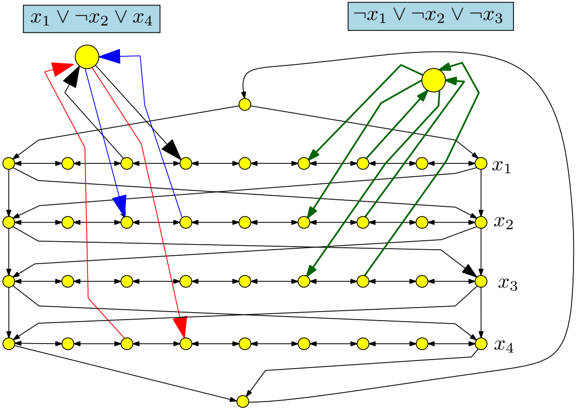


23.3.3
If there is a satisfying assignment, then there is a Hamiltonian cycle

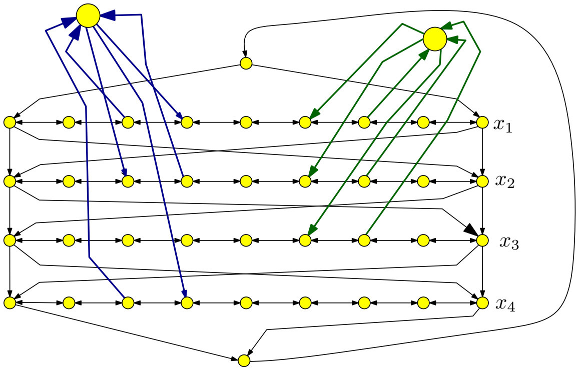
From satisfying assignment to Hamiltonian cycle: By figure

3SAT formula φ:

$$\varphi = \left(\mathbf{x_1} \vee \neg \mathbf{x_2} \vee \mathbf{x_4}\right) \wedge \left(\neg \mathbf{x_1} \vee \neg \mathbf{x_2} \vee \neg \mathbf{x_3}\right)$$



From satisfying assignment to Hamiltonian cycle: By figure



3SAT formula φ:

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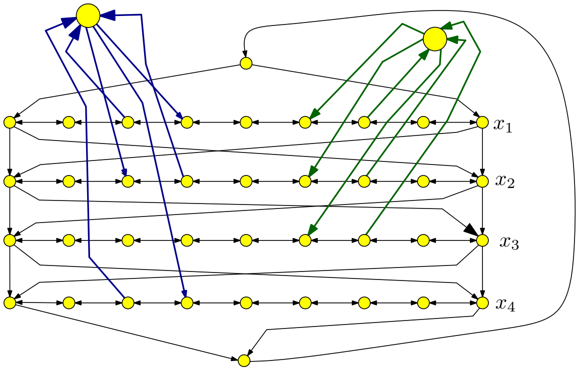
From satisfying assignment to Hamiltonian cycle: By figure



3SAT formula φ:

$$\varphi = \Big(\mathbf{x_1} \vee \neg \mathbf{x_2} \vee \mathbf{x_4}\Big) \wedge \Big(\neg \mathbf{x_1} \vee \neg \mathbf{x_2} \vee \neg \mathbf{x_3}\Big)$$

A satisfying assignment:
$\mathbf{x_1 = 0}$, $\mathbf{x_2 = 1}$, $\mathbf{x_3 = 0}$, $\mathbf{x_4 = 1}$

From satisfying assignment to Hamiltonian cycle: By figure

3SAT formula φ:

$$\varphi = \left(\mathbf{x_1} \vee \neg \mathbf{x_2} \vee \mathbf{x_4}\right) \wedge \left(\neg \mathbf{x_1} \vee \neg \mathbf{x_2} \vee \neg \mathbf{x_3}\right)$$

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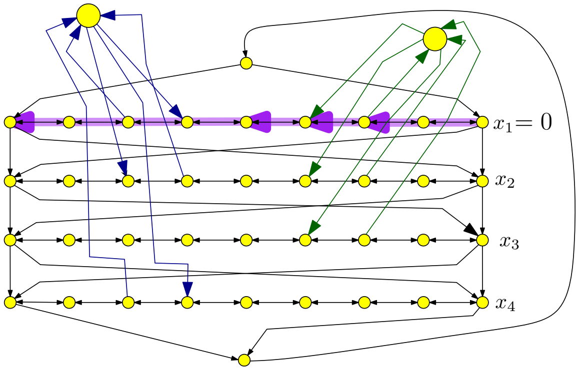

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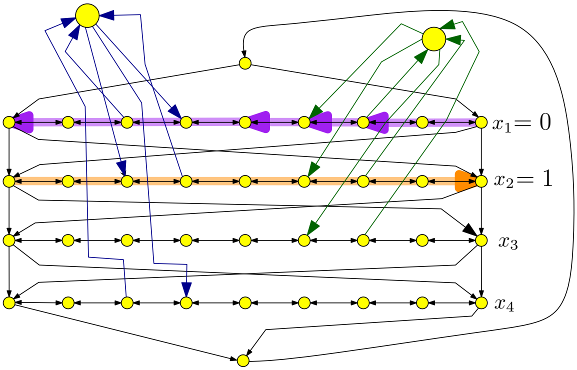

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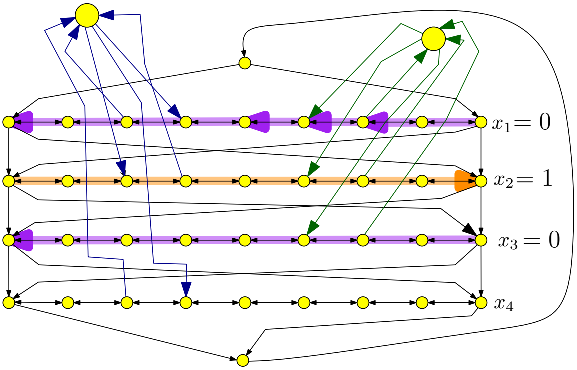

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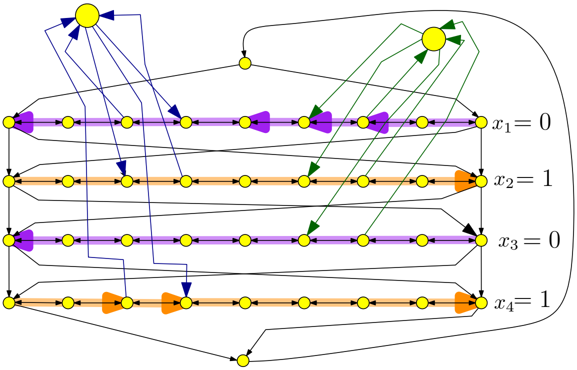

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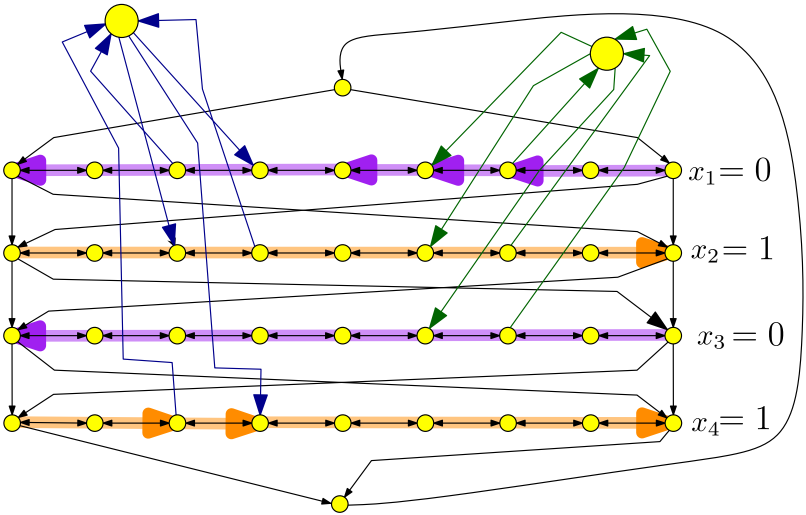
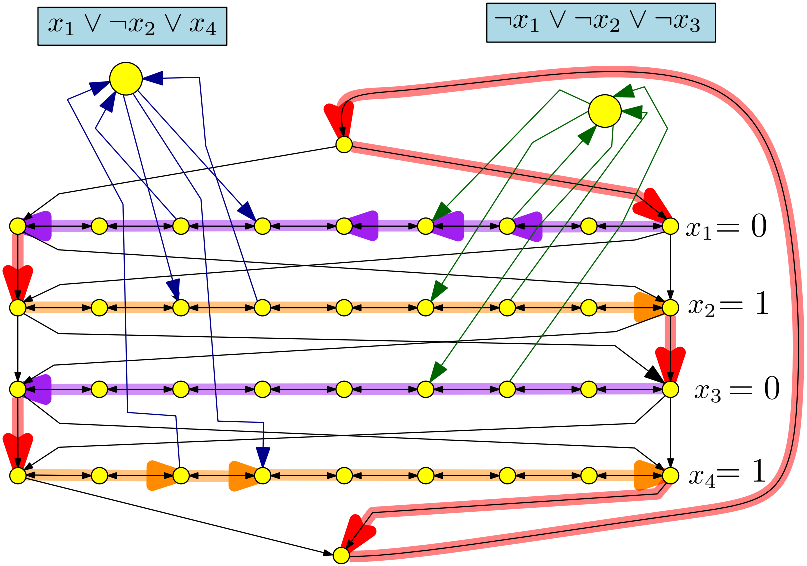
A satisfying assignment:
$\mathbf{x_1 = 0}$, $\mathbf{x_2 = 1}$, $\mathbf{x_3 = 0}$, $\mathbf{x_4 = 1}$

Reduction: Satisfying assignment ⇒ Hamiltonian cycle



Satisfying assignment: $\mathbf{x_1 = 0}$, $\mathbf{x_2 = 1}$, $\mathbf{x_3 = 0}$, $\mathbf{x_4 = 1}$

Reduction: Satisfying assignment $\Rightarrow$ Hamiltonian cycle



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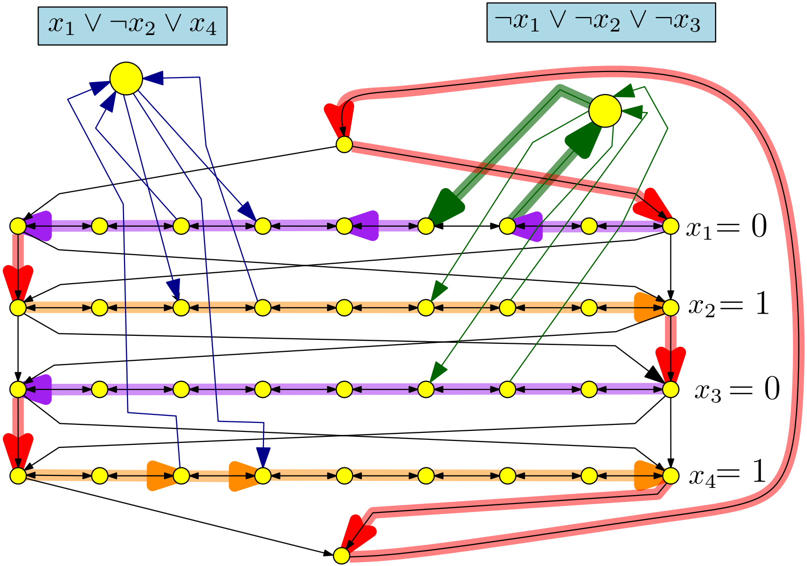
Reduction: Satisfying assignment $\Rightarrow$ Hamiltonian cycle



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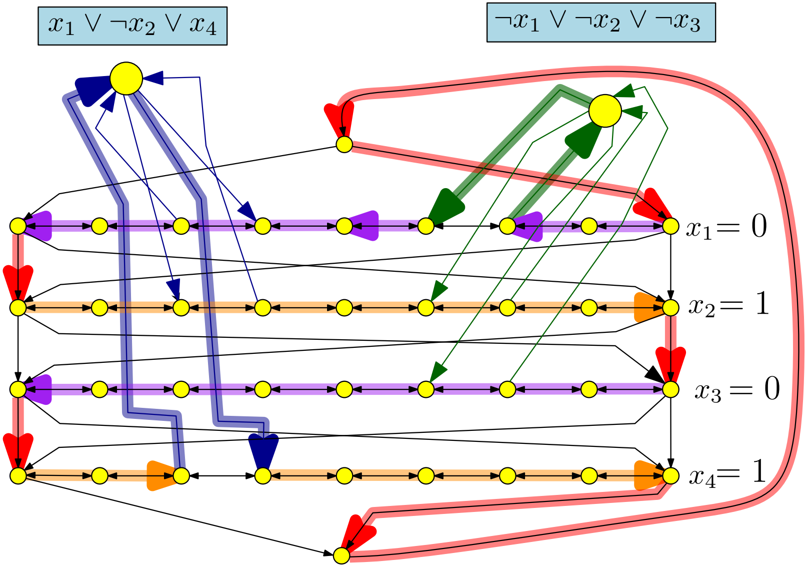

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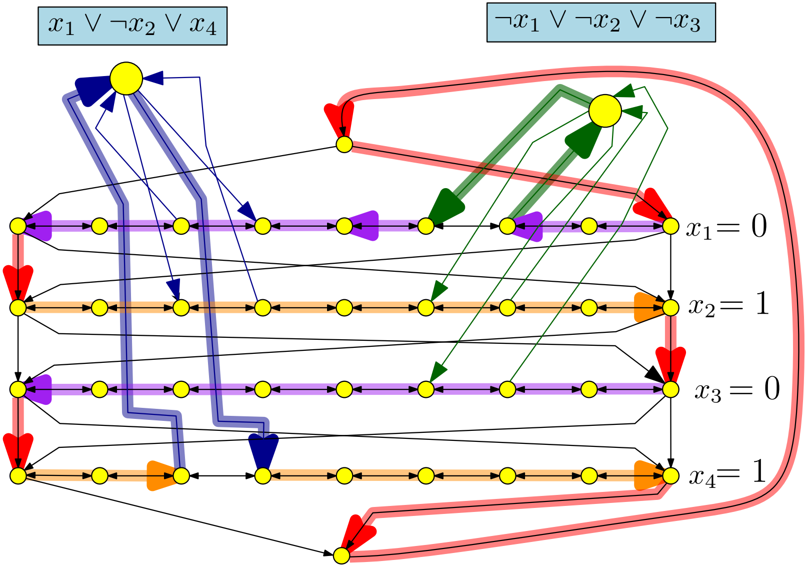
Reduction: Satisfying assignment $\Rightarrow$ Hamiltonian cycle



Satisfying assignment: $\mathbf{x_1 = 0}$, $\mathbf{x_2 = 1}$, $\mathbf{x_3 = 0}$, $\mathbf{x_4 = 1}$

Conclude: If φ has a satisfying assignment then there is an Hamiltonian cycle in G_φ.

Correctness Proof

Lemma 23.1.

φ has a satisfying assignment $\alpha \implies \mathbf{G}_\varphi$ has a Hamiltonian cycle.

Proof.

Let $\mathbf{a}$ be the satisfying assignment for φ. Define Hamiltonian cycle as follows

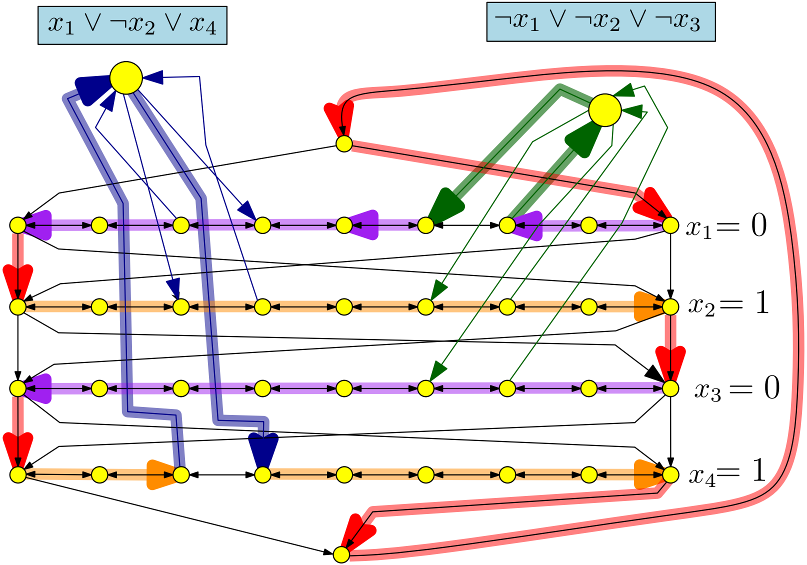
- If $\alpha(\mathbf{x_i}) = \mathbf{1}$ then traverse path $\mathbf{i}$ from left to right
- If $\alpha(\mathbf{x_i}) = \mathbf{0}$ then traverse path $\mathbf{i}$ from right to left
- For each clause, path of at least one variable is in the "right" direction to splice in the node corresponding to clause
- Clearly, resulting cycle is Hamiltonian. □

23.3.4

If there is a Hamiltonian cycle $\implies$ $\exists$satisfying assignment

Reduction: Hamiltonian cycle $\implies \exists$ satisfying assignment

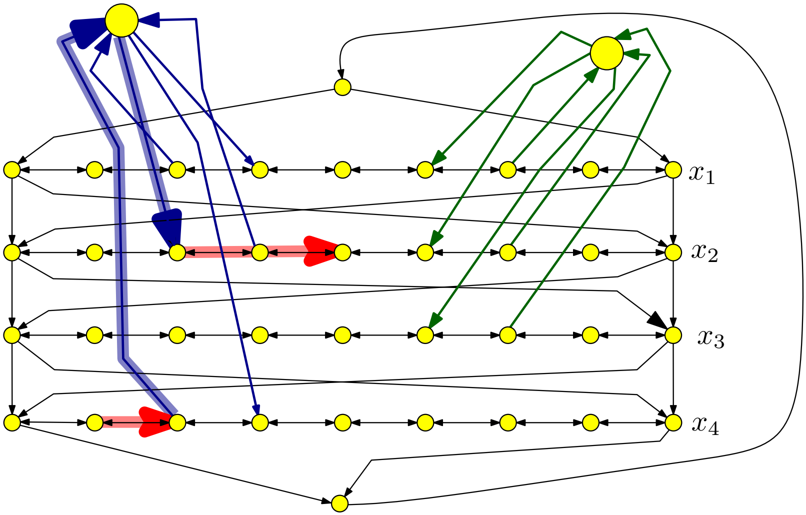
We are given a Hamiltonian cycle in $\mathbf{G}_\varphi$:



Want to extract satisfying assignment...

Reduction: Hamiltonian cycle $\implies \exists$ satisfying assignment

No shenanigan: Hamiltonian cycle can not leave a row in the middle



Reduction: Hamiltonian cycle $\implies \exists$ satisfying assignment

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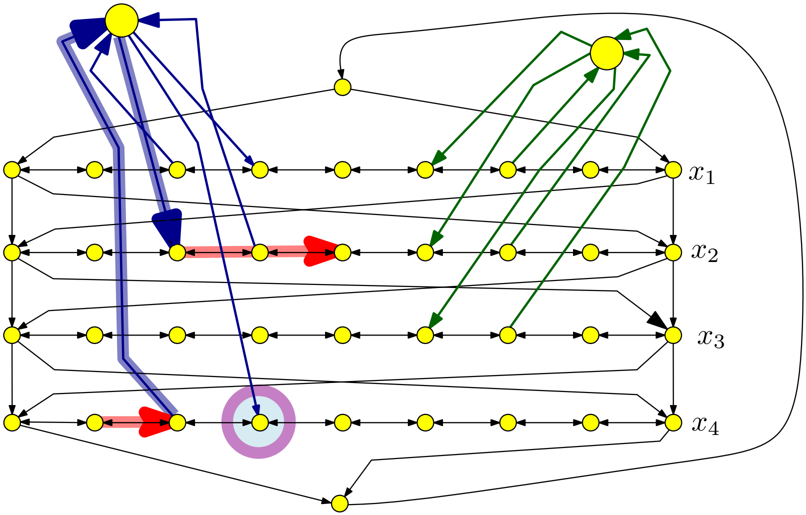

Reduction: Hamiltonian cycle $\implies \exists$ satisfying assignment

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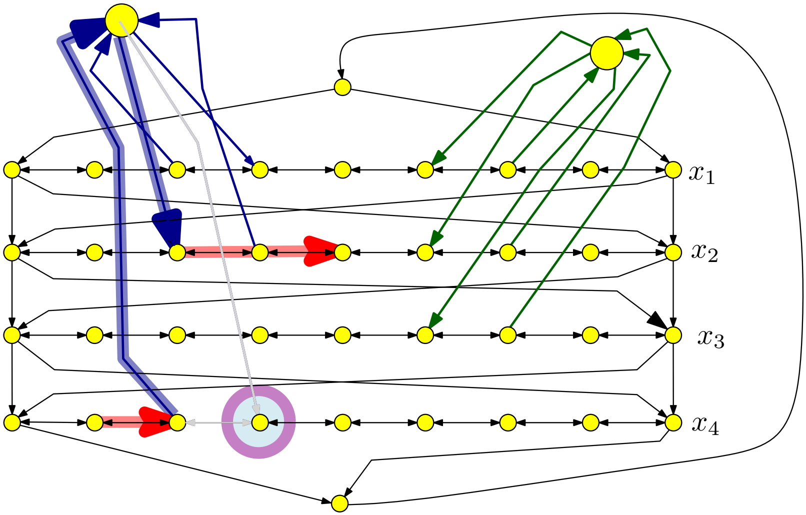
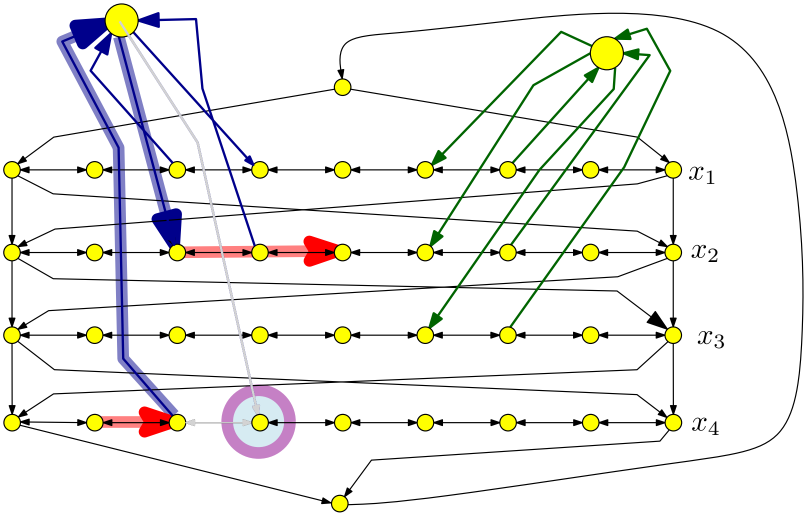


Reduction: Hamiltonian cycle $\implies$ $\exists$ satisfying assignment

No shenanigan: Hamiltonian cycle can not leave a row in the middle



Conclude: Hamiltonian cycle must go through each row completely from left to right, or right to left. As such, can be interpreted as a valid assignment.

Hamiltonian Cycle $\Rightarrow$ Satisfying assignment

Suppose Π is a Hamiltonian cycle in G_φ

- If Π enters c_j (vertex for clause C_j) from vertex $3j$ on path i then it must leave the clause vertex on edge to $3j+1$ on the <u>same path i</u>
  - If not, then only unvisited neighbor of $3j+1$ on path i is $3j+2$
  - Thus, we don't have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle
- Similarly, if Π enters c_j from vertex $3j+1$ on path i then it must leave the clause vertex c_j on edge to $3j$ on path i

Hamiltonian Cycle $\implies$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after $\mathbf{C_i}$ are connected by an edge
- We can remove $\mathbf{c_j}$ from cycle, and get Hamiltonian cycle in $\mathbf{G - c_j}$
- Consider Hamiltonian cycle in $\mathbf{G - \{c_1, \ldots c_m\}}$; it traverses each path in only one direction, which determines the truth assignment

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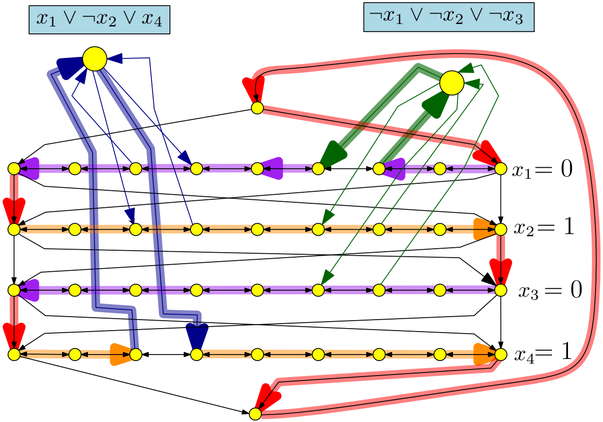

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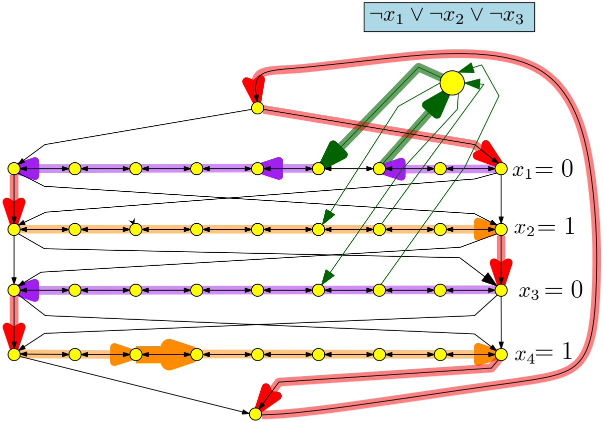

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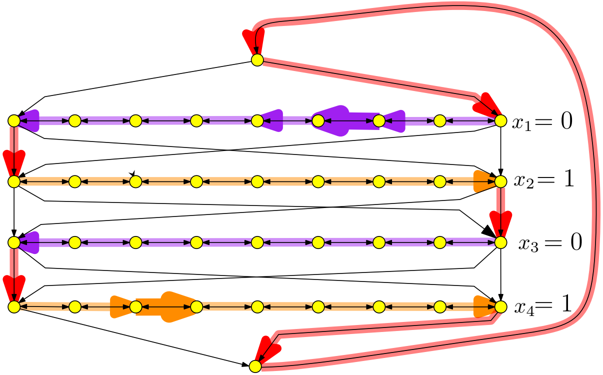

Correctness Proof

We just proved:

Lemma 23.2.

$\mathbf{G}_\varphi$ *has a Hamiltonian cycle* $\Longrightarrow$ φ *has a satisfying assignment* α.

Lemma 23.3.

φ *has a satisfying assignment iff* $\mathbf{G}_\varphi$ *has a Hamiltonian cycle.*

Proof.

Follows from **Lemma 23.1** and **Lemma 23.2** . □

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[redacted]

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Summary

What we did:

1. Showed that **Directed Hamiltonian Cycle** is in **NP**.
2. Provided a polynomial time reduction from **3SAT** to **Directed Hamiltonian Cycle**.
3. Proved that φ satisfiable $\iff$ $\mathbf{G}_\varphi$ is Hamiltonian.

Theorem 23.4.

The problem **Hamiltonian Cycle** *in directed graphs is* **NP-Complete**.

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What we did:

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Theorem 23.4.

The problem **Hamiltonian Cycle** *in directed graphs is* **NP-Complete**.

23.4
Hamiltonian cycle in undirected graph

Hamiltonian Cycle

Problem 23.1.

Input *Given* *undirected* *graph* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$

Goal *Does* $\mathbf{G}$ *have a Hamiltonian cycle? That is, is there a cycle that visits every vertex exactly one (except start and end vertex)?*

NP-Completeness

Theorem 23.2.

Hamiltonian cycle *problem for* **undirected** *graphs is* **NP-Complete**.

Proof.

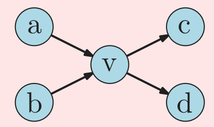
- The problem is in **NP**; proof left as exercise.
- Hardness proved by reducing Directed Hamiltonian Cycle to this problem □

Reduction Sketch

Goal: Given directed graph $\mathbf{G}$, need to construct undirected graph $\mathbf{G}'$ such that $\mathbf{G}$ has Hamiltonian Path iff $\mathbf{G}'$ has Hamiltonian path

Reduction

- Replace each vertex $\mathbf{v}$ by 3 vertices: $\mathbf{v}_{\text{in}}$, $\mathbf{v}$, and $\mathbf{v}_{\text{out}}$
- A directed edge $(\mathbf{a}, \mathbf{b})$ is replaced by edge $(\mathbf{a}_{\text{out}}, \mathbf{b}_{\text{in}})$



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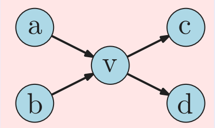

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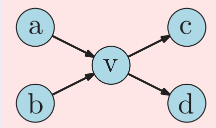

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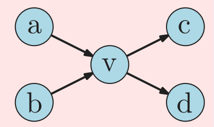
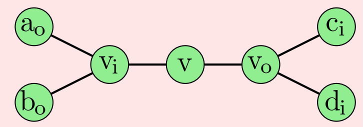
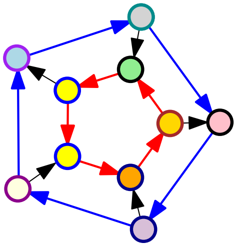


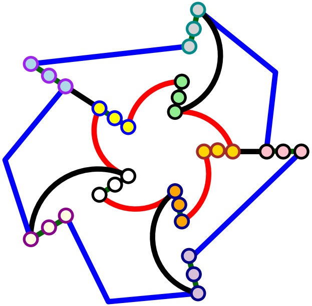


Hamiltonian cycle reduction

Undirected to directed case



$\Rightarrow$



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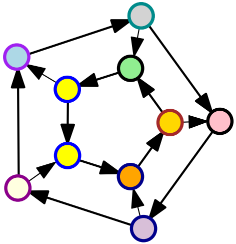

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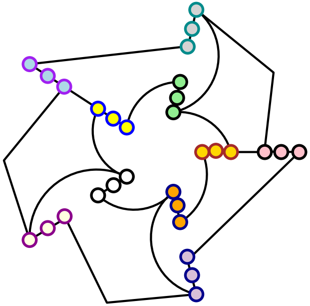

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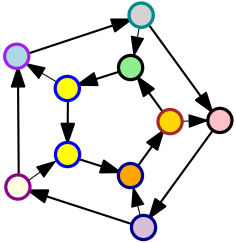

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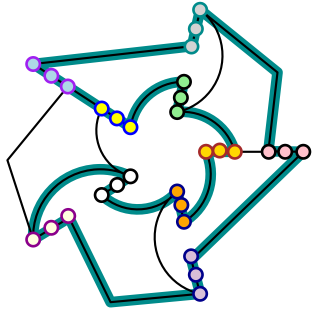

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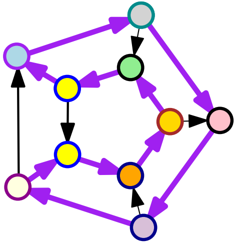

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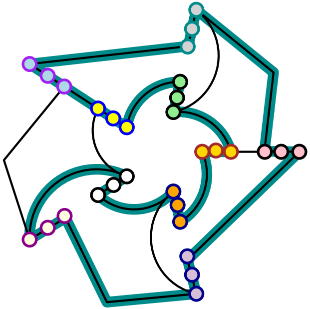

Reduction: Wrap-up

- The reduction is polynomial time (exercise)
- The reduction is correct (exercise)