Deterministic Finite Automata (DFAs)

Lecture 3

Tuesday, September 1, 2020

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3.1
DFA Introduction

DFAs also called Finite State Machines (FSMs)

- The "simplest" model for computers?
- State machines that are common in practice.
  - Vending machines
  - Elevators
  - Digital watches
  - Simple network protocols
- Programs with fixed memory

A simple program

Program to check if a given input string w has odd length

```
int n = 0
While input is not finished
    read next character c
    n ← n + 1
endWhile
If (n is odd) output YES
Else output NO
```

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bit x = 0
While input is not finished
    read next character c
    x ← flip(x)
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If (x = 1) output YES
Else output NO
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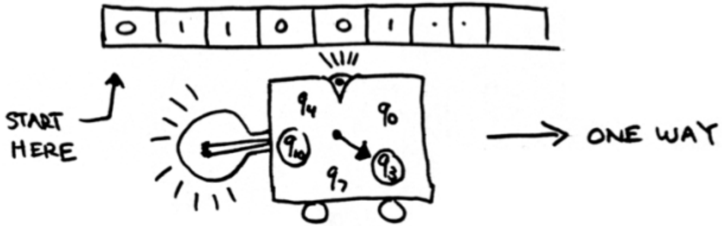
Another view



- Machine has input written on a read-only tape
- Start in specified start state
- Start at left, scan symbol, change state and move right
- Circled states are accepting
- Machine accepts input string if it is in an accepting state after scanning the last symbol.

Draw me a ~~sheep~~ DFA

DFA to check if a given input string has odd length

THE END

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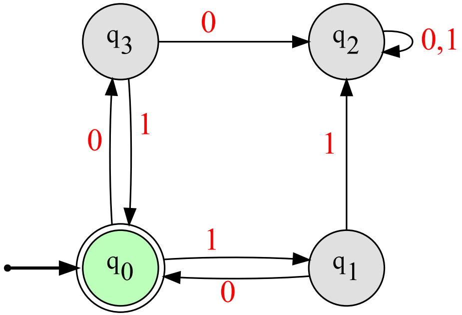
3.1.1

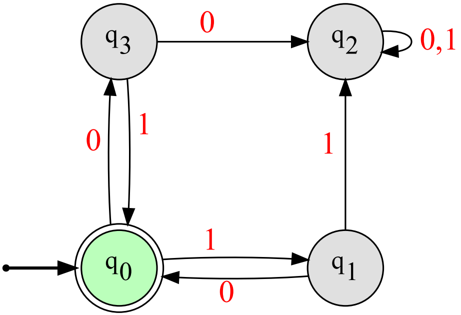
Graphical representation of DFA

Graphical Representation/State Machine



- Directed graph with nodes representing states and edge/arcs representing transitions labeled by symbols in Σ
- For each state (vertex) q and symbol $a \in \Sigma$ there is exactly one outgoing edge labeled by a
- Initial/start state has a pointer (or labeled as s, q_0 or "start")
- Some states with double circles labeled as accepting/final states

Graphical Representation



- Where does 001 lead?
- Where does 10010 lead?
- Which strings end up in accepting state?
- Can you prove it?
- Every string *w* has a unique walk that it follows from a given state *q* by reading one letter of *w* from left to right.

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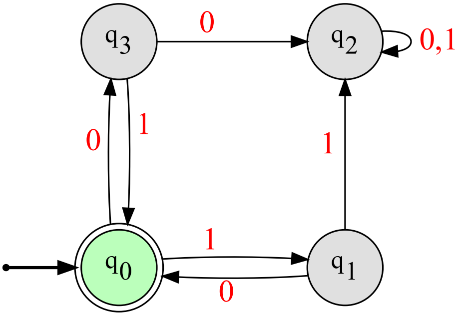

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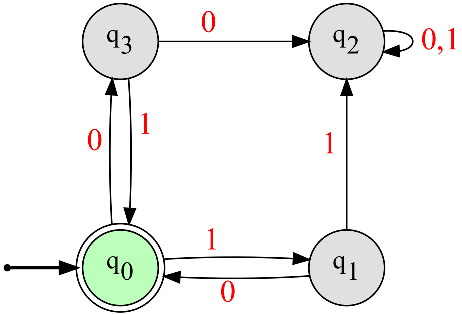

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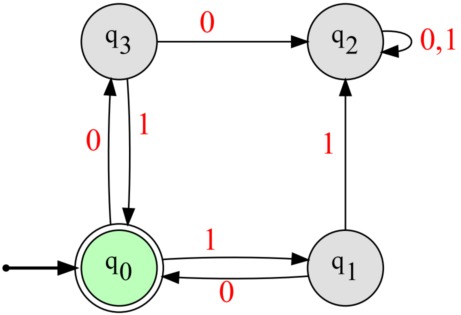

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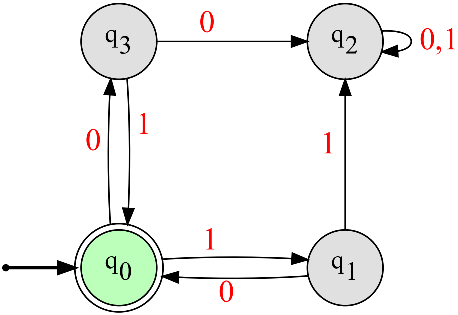

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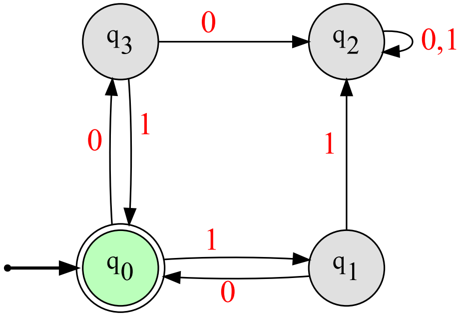

Definition

A DFA M accepts a string w iff the unique walk starting at the start state and spelling out w ends in an accepting state.

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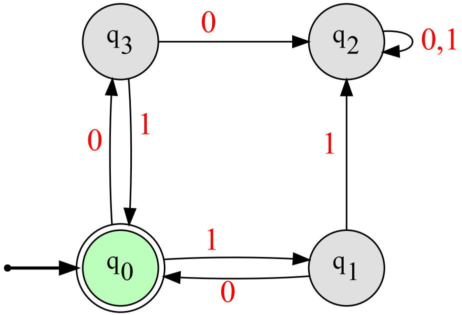

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Definition

The language accepted (or recognized) by a DFA M is denote by $L(M)$ and defined

Warning

"M accepts language L" does not mean simply that that M accepts each string in L.

It means that M accepts each string in L and no others. Equivalently M accepts each string in L and does not accept/rejects strings in $\Sigma^* \setminus L$.

M "recognizes" L is a better term but "accepts" is widely accepted (and recognized) (joke attributed to Lenny Pitt)

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THE END

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3.1.2
Formal definition of DFA

Formal Tuple Notation

Definition

A deterministic finite automata (DFA) $M = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- Q is a finite set whose elements are called states,
- Σ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function,
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

Common alternate notation: q_0 for start state, F for final states.

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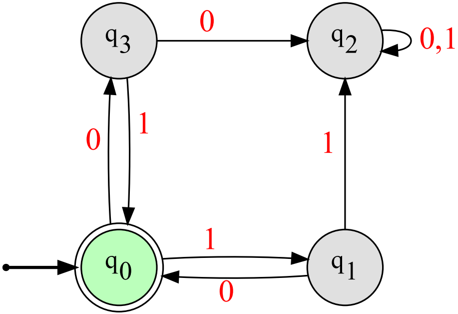
DFA Notation

$$M = \Big(\overbrace{Q}^{\text{set of all states}}, \underbrace{\Sigma}_{\text{alphabet}}, \overbrace{\delta}^{\text{transition func}}, \underbrace{s}_{\text{start state}}, \overbrace{A}^{\text{set of all accept states}} \Big)$$

Example



- $Q = \{q_0, q_1, q_1, q_3\}$
- $\Sigma = \{0, 1\}$
- δ
- $s = q_0$
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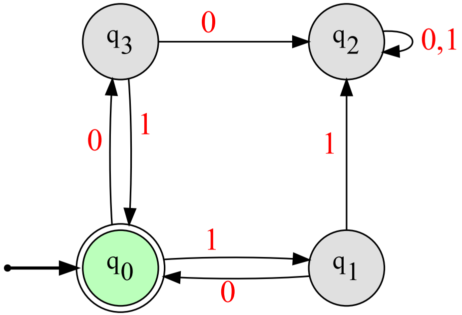

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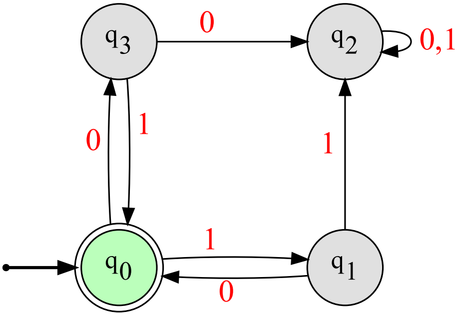

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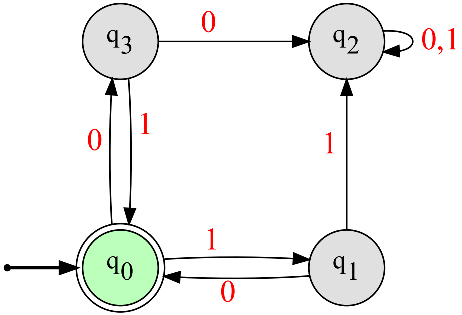

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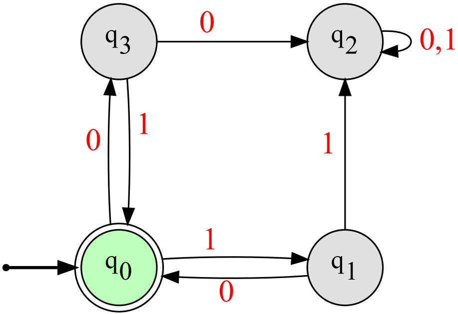

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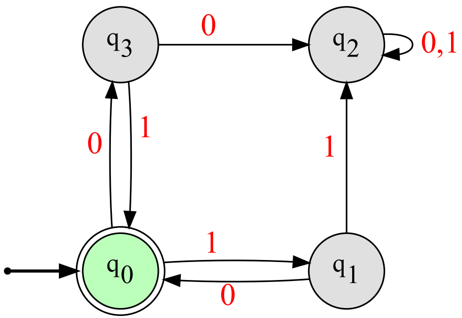

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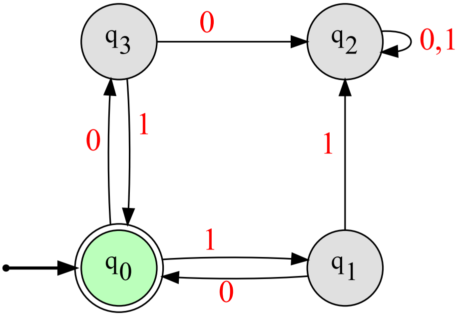

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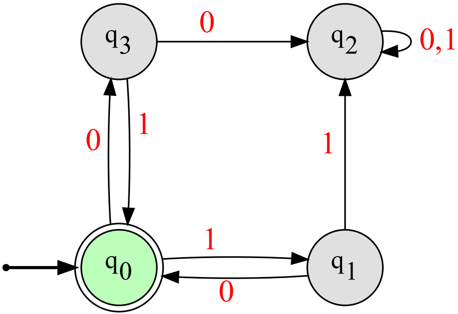

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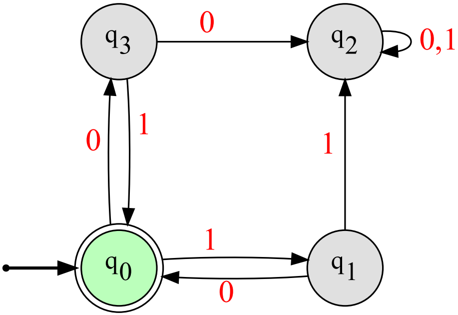

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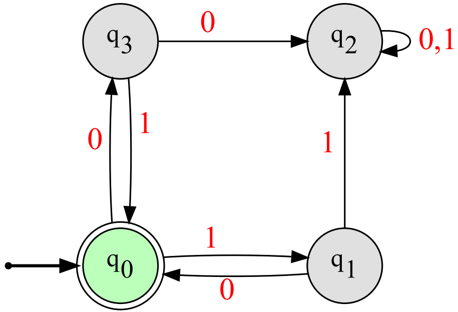
Example: The transition function



δ :

state q	input c	result $\delta(q, c)$
Q	Σ	Σ
q_0	0	q_3
q_0	1	q_1
q_1	0	q_0
q_1	1	q_2
q_2	0	q_2
q_2	1	q_2
q_3	0	q_2
q_3	1	q_0

THE END

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(for now)

3.1.3
Extending the transition function to strings

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Given DFA $M = (Q, \Sigma, \delta, s, A)$, $\delta(q, a)$ is the state that M goes to from q on reading letter a

Useful to have notation to specify the unique state that M will reach from q on reading <u>string</u> w

Transition function $\delta^* : Q \times \Sigma^* \rightarrow Q$ defined inductively as follows:

- $\delta^*(q, w) = q$ if $w = \epsilon$
- $\delta^*(q, w) = \delta^*(\delta(q, a), x)$ if $w = ax$.

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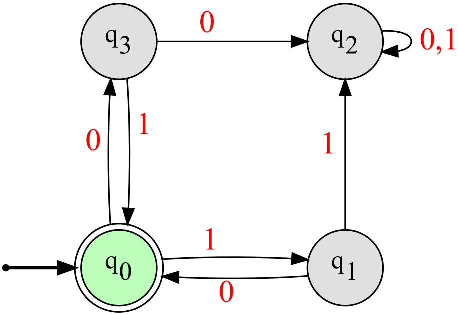
Formal definition of language accepted by M

Definition

The language $L(M)$ accepted by a DFA $M = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \in A\}.$$

Example



What is:

- $\delta^*(q_1, \epsilon)$
- $\delta^*(q_0, 1011)$
- $\delta^*(q_1, 010)$
- $\delta^*(q_4, 10)$
- So what is $L(M)$??????

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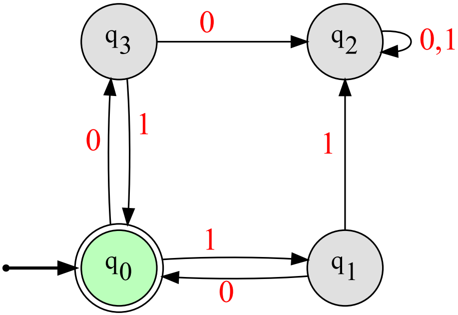

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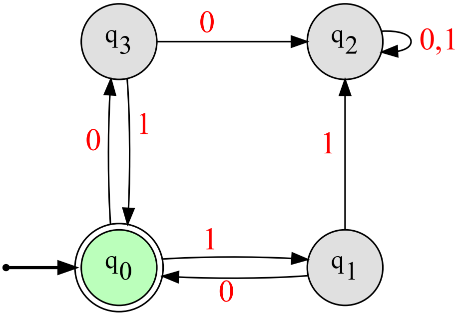

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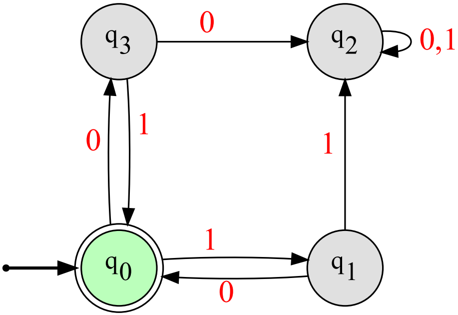

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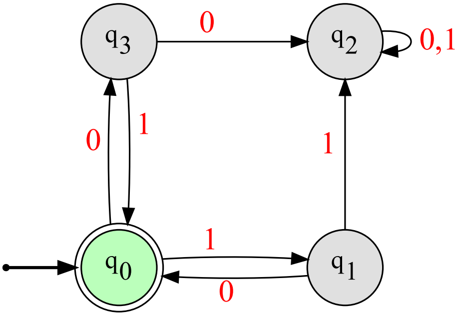

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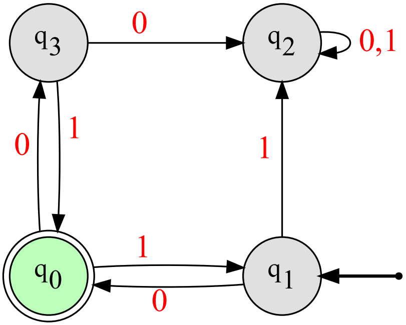

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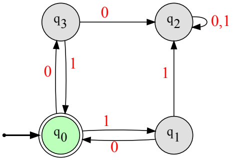
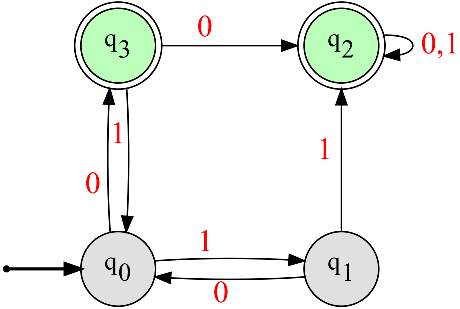


- What is $L(M)$ if start state is changed to q_1?

Example continued



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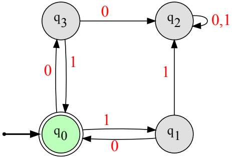

- What is $L(M)$ if final/accept states are set to $\{q_2, q_3\}$ instead of $\{q_0\}$?

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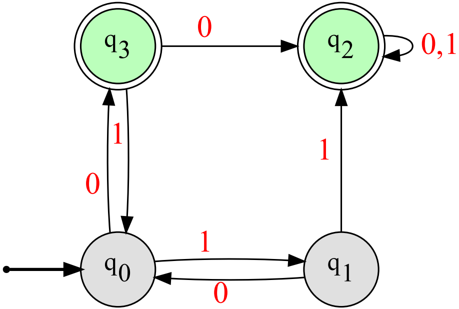
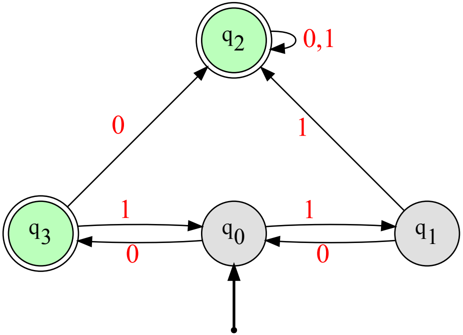


Redraw:



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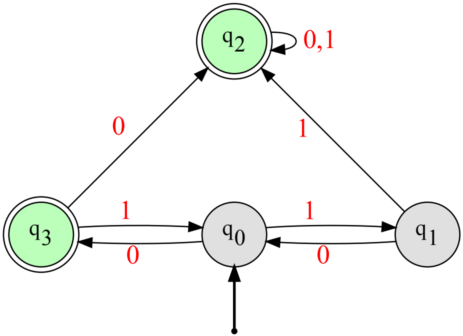

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Advantages of formal specification

- Necessary for proofs
- Necessary to specify abstractly for class of languages

Exercise: Prove by induction that for any two strings u, v, any state q,
$\delta^*(q, uv) = \delta^*(\delta^*(q, u), v)$.

THE END

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(for now)

3.2
Constructing DFAs

DFAs: State = Memory

How do we design a DFA M for a given language L? That is $L(M) = L$.

- DFA is a like a program that has fixed amount of memory independent of input size.
- The memory of a DFA is encoded in its states
- The state/memory must capture enough information from the input seen so far that it is sufficient for the suffix that is yet to be seen (note that DFA cannot go back)

DFA Construction: Examples

Example I: Basic languages

Assume $\Sigma = \{0, 1\}$.
$L = \emptyset$, $L = \Sigma^*$, $L = \{\epsilon\}$, $L = \{0\}$.

DFA Construction: Examples

Example II: Length divisible by 5

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0, 1\}^* \mid |w| \text{ is divisible by } 5\}$

DFA Construction: examples

Example III: Ends with 01

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0,1\}^* \mid w \text{ ends with } 01\}$

DFA Construction: examples

Example IV: Contains 001

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0, 1\}^* \mid w \text{ contains 001 as substring}\}$

DFA Construction: examples

Example V: Contains 001 or 010

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0, 1\}^* \mid w \text{ contains 001 or 010 as substring}\}$

DFA construction examples

Example VI: Has a 1 exactly k positions from end

Assume $\Sigma = \{0, 1\}$.

$L = \{w \mid w \text{ has a } 1 \; k \text{ positions from the end}\}$.

DFA Construction: Example

$L = \{\text{Binary numbers congruent to } 0 \mod 5\}$

Example:

1. $1101011_2 = 107_{10} = 2 \mod 5$,
2. $1010_2 = 10 = 0 \mod 5$

Key observation:

$\mathrm{val}(w) \mod 5 = a$ implies

$$\mathrm{val}(w0) \mod 5 = (\mathrm{val}(w) * 2) \mod 5 = 2a \mod 5$$

$$\mathrm{val}(w1) \mod 5 = (\mathrm{val}(w) \cdot 2 + 1) \mod 5 = (2a + 1) \mod 5$$

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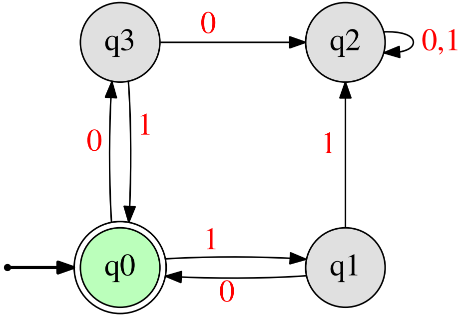
3.3
Complement language

Complement

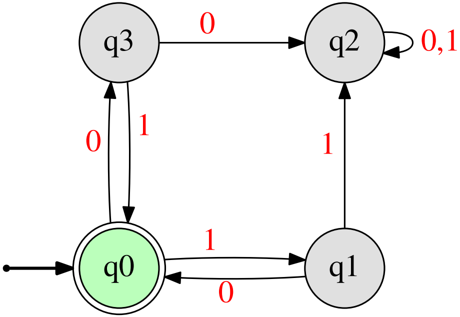
Question: If M is a DFA, is there a DFA M' such that $L(M') = \Sigma^* \setminus L(M)$? That is, are languages recognized by DFAs closed under complement?

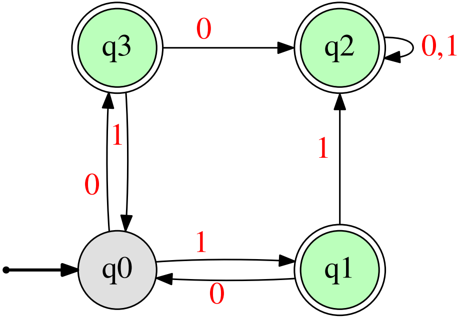


Complement

Example...

Just flip the state of the states!





Complement

Theorem

Languages accepted by DFA*s are closed under complement.*

Proof.

Let $M = (Q, \Sigma, \delta, s, A)$ such that $L = L(M)$.
Let $M' = (Q, \Sigma, \delta, s, Q \setminus A)$. Claim: $L(M') = \bar{L}$. Why?
$\delta^*_M = \delta^*_{M'}$. Thus, for every string w, $\delta^*_M(s, w) = \delta^*_{M'}(s, w)$.
$\delta^*_M(s, w) \in A \Rightarrow \delta^*_{M'}(s, w) \notin Q \setminus A$. $\delta^*_M(s, w) \notin A \Rightarrow \delta^*_{M'}(s, w) \in Q \setminus A$. $\square$

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THE END

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(for now)

3.4
Product Construction

Union and Intersection

Question: Are languages accepted by DFAs closed under union? That is, given DFAs M_1 and M_2 is there a DFA that accepts $L(M_1) \cup L(M_2)$?
How about intersection $L(M_1) \cap L(M_2)$?

Idea from programming: on input string w

- Simulate M_1 on w
- Simulate M_2 on w
- If both accept than $w \in L(M_1) \cap L(M_2)$. If at least one accepts then $w \in L(M_1) \cup L(M_2)$.
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- Solution: Simulate M_1 and M_2 in parallel by keeping track of states of both machines

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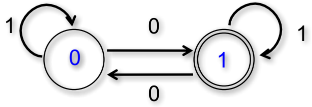
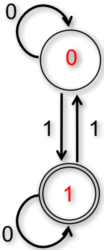


M_1 accepts #0 = odd



M_2 accepts #1 = odd

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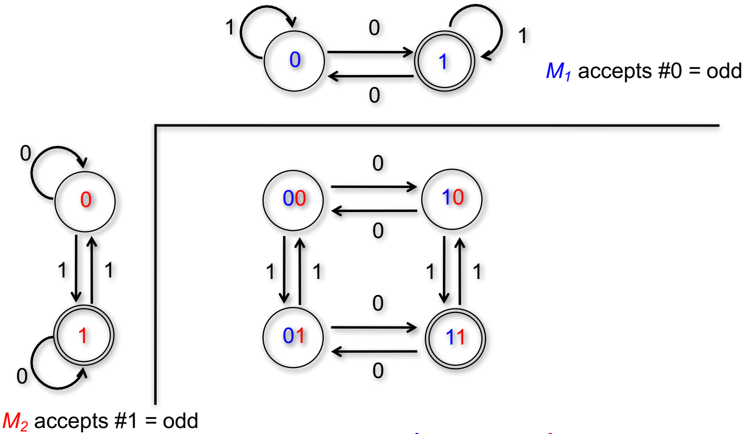
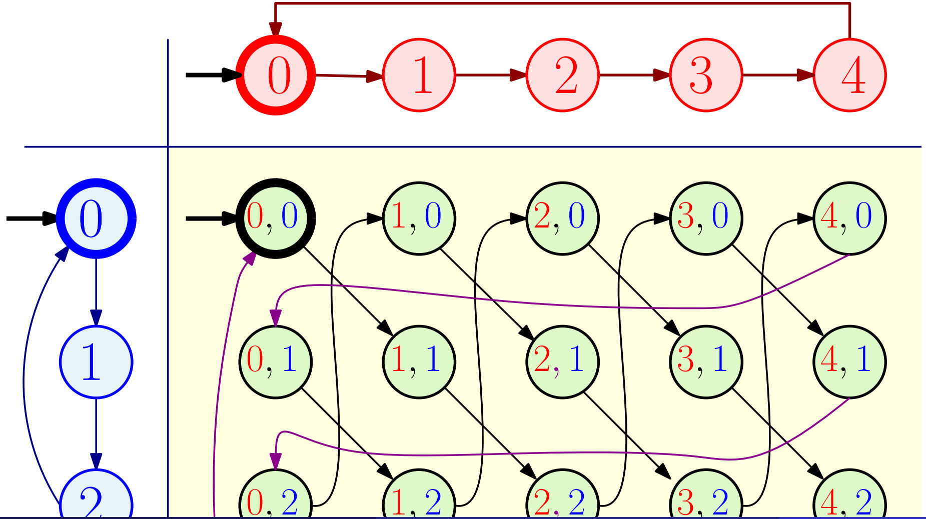


Cross-product machine

Example II

Accept all binary strings of length divisible by 3 and 5



Product construction for intersection

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2)$

Create $M = (Q, \Sigma, \delta, s, A)$ where

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Correctness of construction

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For each string w, $\delta^*(s, w) = (\delta_1^*(s_1, w), \delta_2^*(s_2, w))$.

Exercise: Assuming lemma prove the theorem in previous slide. Proof of lemma by induction on $|w|$

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Set Difference

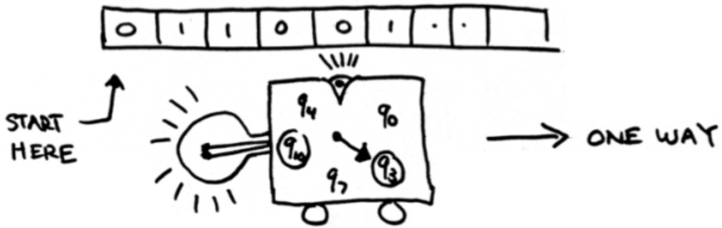
Theorem

M_1, M_2 DFAs. There is a DFA M such that $L(M) = L(M_1) \setminus L(M_2)$.

Exercise: Prove the above using two methods.

- Using a direct product construction
- Using closure under complement and intersection and union

Things to know: 2-way DFA



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Can we allow DFA to scan back and forth? Caveat: Tape is read-only so only memory is in machine's state.

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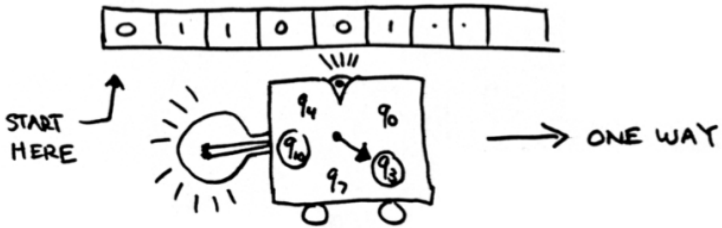

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THE END

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(for now)

3.5
Supplemental: DFA philosophy

A finite program can be simulated by a DFA...

1. Finite program = a program that uses a prespecified bounded amount of memory.
2. Given DFA and input, easy to decide if DFA accepts input.
3. A finite program is a DFA!
   # of states of memory of a finite program = finite.
   $\#$ states $\approx 2^{\#\text{ of memory bits used by program}}$
4. Program using $1K$ memory = has...
5. Turing halting theorem: Not possible (in general) to decide if a program stops on an input.
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So what is going on...

1. Theory models the world. (Oversimplifies it.)
2. Make it possible to think about it.
3. There are cases where theory does not model the world well.
4. Know when to apply the theory.
5. Reject statements that are correct but not useful.
6. Really Large finite numbers are

THE END

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(for now)