7.8

Supplemental: Why $a^n b^n c^n$ is not CFL

You are bound to repeat yourself...

$L = \{a^n b^n c^n \mid n \geq 0\}$.

1. For the sake of contradiction assume that there exists a grammar: G a CFG for L.
2. T_i: minimal parse tree in G for $a^i b^i c^i$.
3. $h_i = \mathrm{height}(T_i)$: Length of longest path from root to leaf in T_i.
4. For any integer t, there must exist an index $j(t)$, such that $h_{j(t)} > t$.
5. There an index j, such that $h_j > \left(2 * \ \# \text{ variables in } G\right)$.

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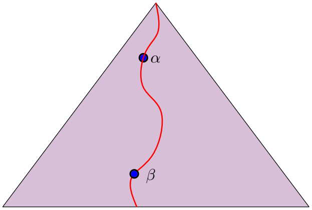
Repetition in the parse tree...



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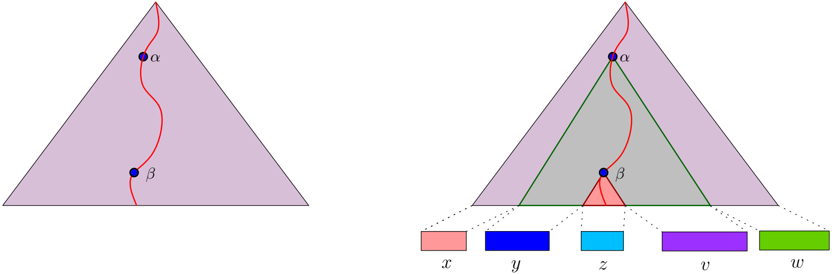

$$xyzvw = a^j b^j c^j$$

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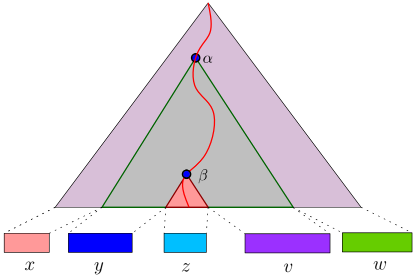

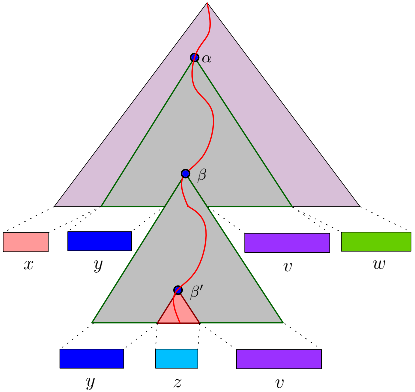

$$xyzvw = a^j b^j c^j \implies xy^2zv^2w \in L$$

- We know:
  $xyzvw = a^i b^j c^j$
  $|y| + |v| > 0$.
- We proved that $\tau = xy^2zv^2w \in L$.
- If y contains both a and b, then, $\tau = ...a...b...a...b...$.
  Impossible, since $\tau \in L = \{a^n b^n c^n \mid n \geq 0\}$.
- Similarly, not possible that y contains both b and c.
- Similarly, not possible that v contains both a and b.
- Similarly, not possible that v contains both b and c.
- If y contains only as, and v contains only bs, then... $\#_{(a)}(\tau) \neq \#_{(c)}(\tau)$.
  Not possible.
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- Must be that $\tau \notin L$. A contradiction.

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We conclude...

Lemma

The language $L = \{a^n b^n c^n \mid n \geq 0\}$ *is not* CFL *(i.e., there is no* CFG *for it).*