“It’s straightforward to link a number to the outcome of an experiment. The result is a Random variable.” ---Prof. Forsythe

Random variable is a function, it is not the same as in $X = X + 1$.

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Random numbers

- Amount of money on a bet
- Age at retirement of a population
- Rate of vehicles passing by the toll
- Body temperature of a puppy in its pet clinic
- Level of the intensity of pain in a toothache
Random variable as vectors

Brain imaging of Human emotions
A) Moral conflict
B) Multi-task
C) Rest

Random Variable
Probability distribution
Cumulative distribution
Joint probability
Independence of random variables
Random variables

\[ X = \begin{cases} 
0 \\
1 
\end{cases} \]
Random variables

-The values of a random variable can be either *discrete*, *continuous* or *mixed*.
The range of a discrete random variable is a countable set of real numbers.
Random Variable Example

- **Number of pairs in a hand of 5 cards**

- Let a single outcome be the hand of 5 cards
- Each outcome maps to values in the set of numbers \( \{0, 1, 2\} \)
Random Variable Example

- **Number of pairs in a hand of 6 cards**
- Let a single outcome be the hand of 6 cards
- What is the range of values of this random variable?
If we roll a 3-sided fair die, and define random variable $U$, such that

A. $\{-1, 0, 1\}$

B. $\{0, 1\}$
Three important facts of Random variables

- Random variables have probability functions
- Random variables can be conditioned on events or other random variables
- Random variables have averages
Random variables have probability functions

- Let $X$ be a random variable
- The set of outcomes is an event with probability

$$P(X = x_0)$$

$X$ is the random variable
$x_0$ is any unique instance that $X$ takes on
\( P(X = x) \) is called the probability distribution for all possible \( x \).

\( P(X = x) \) is also denoted as \( P(x) \) or \( p(x) \).

\( P(X = x) \geq 0 \) for all values that \( X \) can take, and is 0 everywhere else.

The sum of the probability distribution is 1:
\[
\sum_{x} P(x) = 1
\]
Cumulative distribution

- $P(X \leq x)$ is called the cumulative distribution function of $X$
- $P(X \leq x)$ is also denoted as $f(x)$
- $P(X \leq x)$ is a non-decreasing function of $x$
Probability distribution and cumulative distribution

 ramifications

 Give the random variable $X$,

 $X(\omega) = \begin{cases} 
 1 & \text{outcome of } \omega \text{ is head} \\
 0 & \text{outcome of } \omega \text{ is tail}
 \end{cases}$

 $p(x) \quad P(X = x)$

 $f(x) \quad P(X \leq x)$

 $1/2 \quad 1$
Function of random variables: die example

Roll 4-sided fair die twice.

Define these random variables:

- $X$, the values of 1\textsuperscript{st} roll
- $Y$, the values of 2\textsuperscript{nd} roll

Sum $S = X + Y$

Difference $D = X - Y$

Size of Sample Space = ?
Random variable: die example

\[ S = X + Y \]

\[ D = X - Y \]

\[
P(S = 7)
\]

\[
P(D \leq -1)
\]
Probability distribution of the sum of two random variables

• Give the random variable $S$ in the 4-sided die, whose range is $\{2,3,4,5,6,7,8\}$, probability distribution of $S$.
Give the random variable $D = X - Y$, what is the probability distribution of $D$?
Conditional Probability

The probability of $A$ given $B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(B) \neq 0$

The "Size" analogy

$$\sum_x P(x|y) = 1$$

Credit: Prof. Jeremy Orloff & Jonathan Bloom
The conditional probability distribution of $X$ given $Y$ is

$$P(x|y) = \frac{P(x, y)}{P(y)} \quad P(y) \neq 0$$
Conditional probability distribution of random variables

The conditional probability distribution of $X$ given $Y$ is

$$P(x|y) = \frac{P(x, y)}{P(y)} \quad P(y) \neq 0$$

The joint probability distribution of two random variables $X$ and $Y$ is

$$P(\{X = x\} \cap \{Y = y\})$$

$$\sum_x P(x|y) = 1$$
Get the marginal from joint distri.

We can recover the individual probability distributions from the joint probability distribution

\[ P(x) = \sum_y P(x, y) \]

\[ P(y) = \sum_x P(x, y) \]
Joint probabilities sum to 1

The sum of the joint probability distribution

\[ \sum_{x} \sum_{y} P(x, y) = 1 \]
Joint Probability Example

Tossing a coin twice, we define random variable $X$ and $Y$ for each toss.

$X(\omega) = \begin{cases} 
1 & \text{outcome of } \omega \text{ is head} \\
0 & \text{outcome of } \omega \text{ is tail} 
\end{cases}$

$Y(\omega) = \begin{cases} 
1 & \text{outcome of } \omega \text{ is head} \\
0 & \text{outcome of } \omega \text{ is tail} 
\end{cases}$
Joint probability distribution example

\[ P(x, y) \]

\begin{array}{cc}
0 & 1 \\
\hline
Y \downarrow & X \uparrow & \hline
0 & 1 \\
\hline
P(x) & P(y) \\
\end{array}
Joint Probability Example

Now we define Sum $S = X + Y$, Difference $D = X - Y$. $S$ takes on values \{0,1,2\} and $D$ takes on values \{-1, 0, 1\}

\[
X(\omega) = \begin{cases} 
1 & \text{outcome of } \omega \text{ is head} \\
0 & \text{outcome of } \omega \text{ is tail}
\end{cases}
\]

\[
Y(\omega) = \begin{cases} 
1 & \text{outcome of } \omega \text{ is head} \\
0 & \text{outcome of } \omega \text{ is tail}
\end{cases}
\]
Joint Probability Example

Suppose coin is fair, and the tosses are independent.
Joint probability distribution example

\[ P(s, d) \]

\[
\begin{array}{ccc}
    & -1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
\end{array}
\]

\[ D \]

\[ P(s) \]

\[ P(d) \]
Independence of random variables

随机变量 $X$ 和 $Y$ 是独立的，如果

$$P(x, y) = P(x)P(y) \quad \text{for all } x \text{ and } y$$

在之前的硬币抛掷示例中

- $X$ 和 $Y$ 独立吗？
- $S$ 和 $D$ 独立吗？
Joint probability distribution example

\[ P(x, y) \]

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>1</td>
<td>1/4</td>
<td>1/4</td>
</tr>
</tbody>
</table>

\[ P(x) \]

|   | 1/2 | 1/2 |

\[ P(y) \]

|   | 1/2 | 1/2 |

\[ X \]

\[ Y \]

\[ P(x, y) \]
Joint probability distribution example

\[ P(s, d) \]

\[
\begin{array}{ccc}
S & -1 & 0 & 1 \\
0 & 0 & \frac{1}{4} & 0 \\
1 & \frac{1}{4} & 0 & \frac{1}{4} \\
2 & 0 & \frac{1}{4} & 0 \\
\end{array}
\]

\[ D \]

\[ P(s) \]

\[
\begin{array}{c}
\frac{1}{4} \\
\frac{1}{2} \\
\frac{1}{4} \\
\end{array}
\]

\[ P(d) \]

\[
\begin{array}{ccc}
P & 0 & 1 & 2 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\end{array}
\]
Conditional probability distribution example

$$P(s|d) = \frac{P(s, d)}{P(d)}$$

$S$

$\begin{array}{c|ccc}
\hline
D & -1 & 0 & 1 \\
\hline
0 & 0 & 1/2 & 0 \\
1 & 1 & 0 & 1 \\
2 & 0 & 1/2 & 0 \\
\hline
\end{array}$

$P(s, d)$

$P(d)$
Bayes rule for events generalizes to random variables

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

\[ P(x|y) = \frac{P(y|x)P(x)}{P(y)} \]

\[ = \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)} \]

Total Probability
Conditional probability distribution example

\[ P(s|d) = \frac{P(s, d)}{P(d)} \]

\[
\begin{array}{cccc}
0 & 1 & 0 & 1 \\
\hline
0 & 0 & \frac{1}{2} & 0 \\
1 & 1 & 0 & 1 \\
2 & 0 & \frac{1}{2} & 0 \\
\end{array}
\]

\[
P(D = -1|S = 1) = \frac{P(S = 1|D = -1)P(D = -1)}{P(S = 1)} = \frac{1 \times \frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}
\]
Additional References

* Charles M. Grinstead and J. Laurie Snell
  "Introduction to Probability"

* Morris H. Degroot and Mark J. Schervish
  "Probability and Statistics"
See you next time

See You!