“It’s straightforward to link a number to the outcome of an experiment. The result is a Random variable.” ---Prof. Forsythe

Random variable is a function, it is not the same as in $X = X+1$
Conditional Probability

Review

Total probability

Independence

\[
P(A \cap B) = P(A) \quad \text{and} \quad P(A \cap B) = P(A)P(B)
\]

Bayes

\[
P(A | B) = \frac{P(A \cap B)}{P(B)}
\]

\[
P(C) = \sum P(B | A_i)P(A_i)
\]

Ai \cap A_j = \emptyset \quad \forall A_i \neq j

\[\cup A_i = B\]
Independence of empty event

Q. Any event is independent of empty event B.

A. True
B. False

\[
P(A \cap B) = P(A)P(B)
\]

\[
P(B) = 0
\]

\[
A \cap \emptyset = \emptyset
\]
Which is larger?

1. The probability of drawing hands of 5-cards that have no pairs.
   (no replacement)

2. 0.5

A. 0
B. 2

\[ E: \text{no pairs} = \binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{52 	imes 51 	imes 50 	imes 49 	imes 48}{5 	imes 4 	imes 3 	imes 2 	imes 1} = 2,
\]

\[ P(E) = \frac{\binom{52}{5}}{\binom{52}{5}} = 0.507 \]
Do not consider order.

\[ |E_1| = \binom{13}{5} \cdot 4^5 \]

\[ |S_1| = \binom{52}{5} \]
Random numbers

- Amount of money on a bet
- Age at retirement of a population
- Rate of vehicles passing by the toll
- Body temperature of a puppy in its pet clinic
- Level of the intensity of pain in a toothache
Random variable as vectors

Brain imaging of Human emotions
A) Moral conflict
B) Multi-task
C) Rest

(x, y, t, i)

Objectives

- Random Variable
- Probability distribution
- Cumulative distribution
- Joint probability
- Independence of random variables
A random variable maps all outcomes to numbers, \((w) \rightarrow (x)\). Outcomes are disjoint.

**Bernoulli random variable**

\[ P(w = \text{head}) = p \]
\[ P(w = \text{tail}) = 1 - p \]
Random variables

- The values of a random variable can be either discrete, continuous or mixed.
Discrete Random variables

The range of a discrete random variable is a countable set of real numbers.
Random Variable Example

- Number of pairs in a hand of 5 cards

Let a single outcome be the hand of 5 cards.

Each outcome maps to values in the set of numbers \{0, 1, 2\}.

\( \omega_i = \) 

\( X(\omega_i) = ? = 0 \)

0, 1, 2 are the possible values.
Number of pairs in a hand of 6 cards

Let a single outcome be the hand of 6 cards

What is the range of values of this random variable?

\[ X(w) \text{ could take } [0, 1, 2, 3] \]
If we roll a 3-sided fair die, and define random variable $U$, such that

$U(w) = \begin{cases} 
-1 & w \rightarrow \text{side 1} \\
0 & w \rightarrow \text{side 2} \\
1 & w \rightarrow \text{side 3}
\end{cases}$

$X = U^2$

What is the range of $X$?

$X(w) = \begin{cases}
0 & w \rightarrow \text{side 2} \\
1 & w \rightarrow \text{side 1 or 3}
\end{cases}$

A. $\{-1, 0, 1\}$  
B. $\{0, 1\}$
Three important facts of Random variables

- Random variables have probability functions
- Random variables can be conditioned on events or other random variables
- Random variables have averages
Random variables have probability functions

- Let $X$ be a random variable
- The set of outcomes $\{ \omega \in \mathbb{R}, \text{s.t. } X(\omega) = x_0 \}$ is an event with probability

$$P(X = x_0)$$

$X$ is the random variable
$x_0$ is any unique instance that $X$ takes on
Probability Distribution

• $P(X = x)$ is called the probability distribution for all possible $x$

• $P(X = x)$ is also denoted as $P(x)$ or $p(x)$

• $P(X = x) \geq 0$ for all values that $X$ can take, and is 0 everywhere else

• The sum of the probability distribution is 1

$$\sum_{x} P(x) = 1$$
Examples of Probability Distributions

- $X(\omega) = \begin{cases} 1 & \text{head (fair coin)} \\ 0 & \text{tail} \end{cases}$
- $U(\omega) = \begin{cases} -1 & \text{side 1} \\ 0 & \text{side 2} \\ 1 & \text{side 3} \end{cases}$

- $X(\omega) = U^2$

- $X(\omega) = \begin{cases} 0 & \text{# of pairs = 0} \\ \frac{1}{2} & \text{# pairs in a hand of 5-cards} \\ 1 & \text{# of pairs} \end{cases}$
Another way to write PDF

\[ x(w) = \begin{cases} 
1 & \text{head} \\
0 & \text{tail} 
\end{cases} \]

\[ P(X=x) = \begin{cases} 
\begin{cases} 
\frac{1}{2} & x = 0 \\
\frac{1}{2} & x = 1 \\
0 & \text{otherwise} 
\end{cases} 
\end{cases} \]

\[ U(w) = \begin{cases} 
-1 & \text{side 1} \\
0 & \text{side 2} \\
1 & \text{side 3} 
\end{cases} \]

\[ X(w) = U^2 \]

\[ P(X=x) = \begin{cases} 
\begin{cases} 
\frac{1}{3} & x = 0 \\
\frac{2}{3} & x = 1 \\
0 & \text{otherwise} 
\end{cases} 
\end{cases} \]
Cumulative distribution

- $P(X \leq x)$ is called the cumulative distribution function of $X$
- $P(X \leq x)$ is also denoted as $f(x)$
- $P(X \leq x)$ is a non-decreasing function of $x$

If $x = \max \text{ of range}$, then $p(X \leq x) = 1$ for $x$
Give the random variable $X$, \[ X(\omega) = \begin{cases} 
1 & \text{outcome of } \omega \text{ is head} \\
0 & \text{outcome of } \omega \text{ is tail} 
\end{cases} \]

$p(x)$ is the probability mass function (PMF) of $X$, \[ P(X = x) \]

$f(x)$ is the probability density function (PDF) of $X$, \[ P(X \leq x) \]

CDF
What is the value?

A biased four-sided die is rolled once. Random variable $X$ is defined to be the down-face value.

$$P(X = x) = \begin{cases} \frac{x}{10} & \text{if } x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

$P(X \leq 4)$

A) 0.1
B) 0.3
C) 0.2
D) 0.6
Functions of Random Variables

\[ X = \max \{ U \} \]

\[ X = U_1 + U_2 \]

\[ S = x + y + \ldots \]

\[ D = x - y \]
Q. Are these random variables the same?

\[ X(w) = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases} \quad Y(w) = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases} \]

\[ U = 2X \quad V = X + Y \]

Are \( U \) and \( V \) the same?

A) Yes

B) No.

Whether PDFs are the same?
Function of random variables: die example

Roll 4-sided fair die twice. (randomly)

Define these random variables:

- $X$, the values of 1\textsuperscript{st} roll
- $Y$, the values of 2\textsuperscript{nd} roll

Sum $S = X + Y$
Difference $D = X - Y$

Size of Sample Space = ?
Roll 4-sided fair die twice.

\[ P(X = 1) \]
\[ P(Y \leq 2) \]
\[ P(S = 7) \]
\[ P(D \leq -1) \]

Size of Sample Space = 16
**Random variable: die example**

\[ S = X + Y \]

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<tr>
<th>Y</th>
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<th>3</th>
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\[ D = X - Y \]

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\[ P(S = 7) = \frac{2}{16} = \frac{1}{8} \]

\[ P(D \leq -1) = \frac{6}{16} \]
Give the random variable $S$ in the 4-sided die, whose range is $\{2,3,4,5,6,7,8\}$, probability distribution of $S$. 

$p(s)$

1/16

$S$
Probability distribution of the difference of two random variables

Given the random variable \( D = X - Y \), what is the probability distribution of \( D \)?

![Graph showing the probability distribution of \( D \)]
Assignments

❖ Module Week 4, HW3 due tonight, quiz.

❖ Next time: More random variable, Expectations, Variance
Additional References

- Charles M. Grinstead and J. Laurie Snell
  "Introduction to Probability"

- Morris H. Degroot and Mark J. Schervish
  "Probability and Statistics"
See you next time

See You!