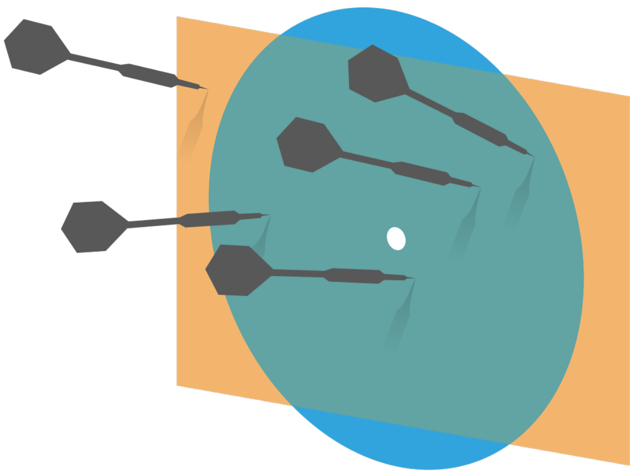
Probability and Statistics for Computer Science



Credit: wikipedia

"It's straightforward to link a number to the outcome of an experiment. The result is a **Random variable.**" ---Prof. Forsythe

Random variable is a function, it is not the same as in **X = X+1**

Last time

- Conditional Probability
  - Review
  - Total probability
  - Independence

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

Bayes

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$$P(B) = \sum P(B|A_i)P(A_i)$$

$$A_i \cap A_j = \phi \quad i \neq j$$

$$\bigcup_i A_i \supset B$$

$$P(A|B) = P(A)$$

$$P(A\cap B) = P(A)P(B)$$

Independence of empty event

- Q. Any event is independent of empty event B.

  $\boxed{\text{A.}}$ True

  B. False

$P(A \cap B) = P(A)\,P(B)$

$\overset{\shortparallel}{0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{0}$

$P(B) = 0$

$A \cap \phi = \phi$

Which is larger?

① The probability of drawing hands of 5- cards that have no pairs.

(no replacement)

② 0.5

$\boxed{A.}$ ①   B. ②

E: no pairs $|E| = 52 \times 48 \times 44 \times 40 \times 36$

Ω: $|\Omega| = {}_{52}P_5 = 52 \times 51 \times 50 \times 49 \times 48$

$$P(E) = \frac{52 \times 48 \times \cdots \times 36}{{}_{52}P_5} = 0.507$$

Do not consider order

Pick # Pick suit

$$|E| = \binom{13}{5} \cdot 4^5$$

$$|\Omega| = \binom{52}{5}$$

Random numbers

- Amount of money on a bet
- Age at retirement of a population
- Rate of vehicles passing by the toll
- Body temperature of a puppy in its pet clinic
- Level of the intensity of pain in a toothache

Random variable as vectors

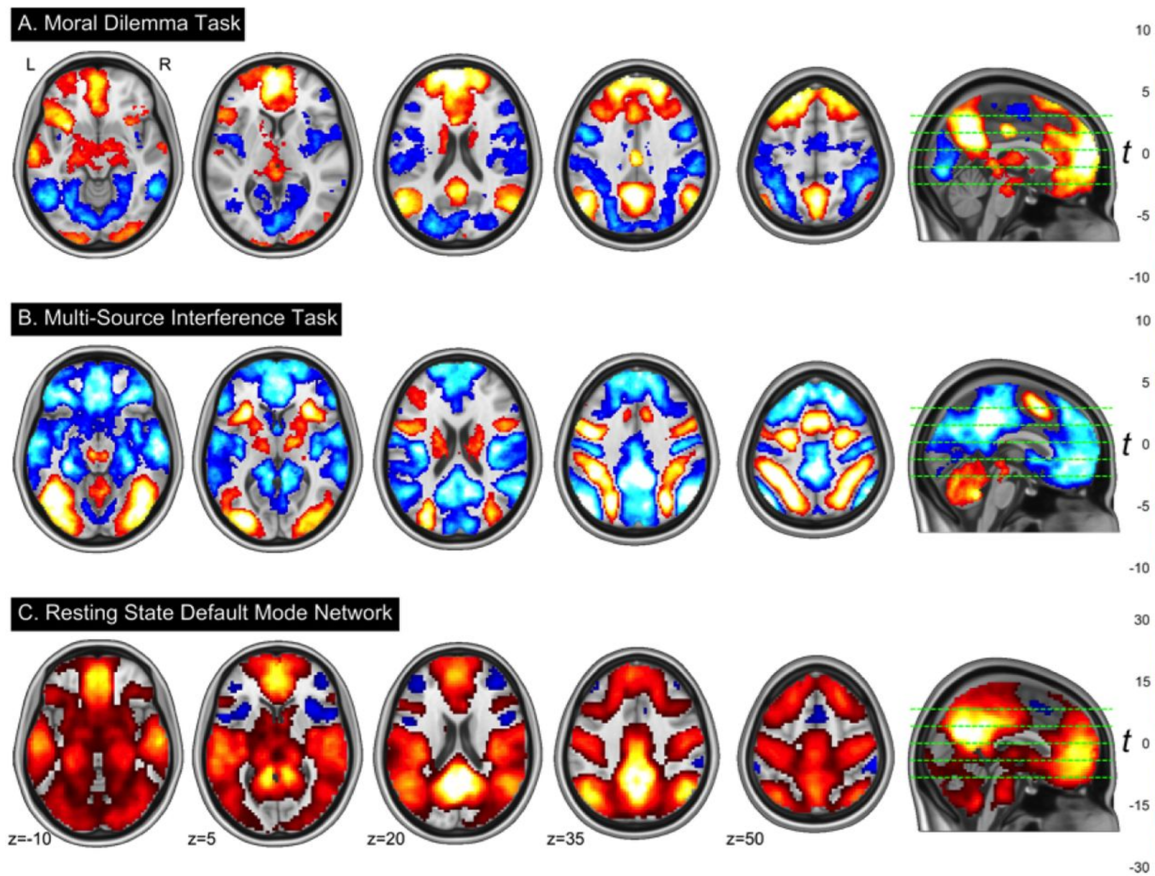
Brain imaging of Human emotions

A) Moral conflict

B) Multi-task

C) Rest

(x, y, z, t, i)



A. McDonald et al. NeuroImage doi: 10.1016/j.neuroimage.2016.10.048

Objectives

- Random Variable
- Probability distribution
- Cumulative distribution
- Joint probability
- Independence of random variables

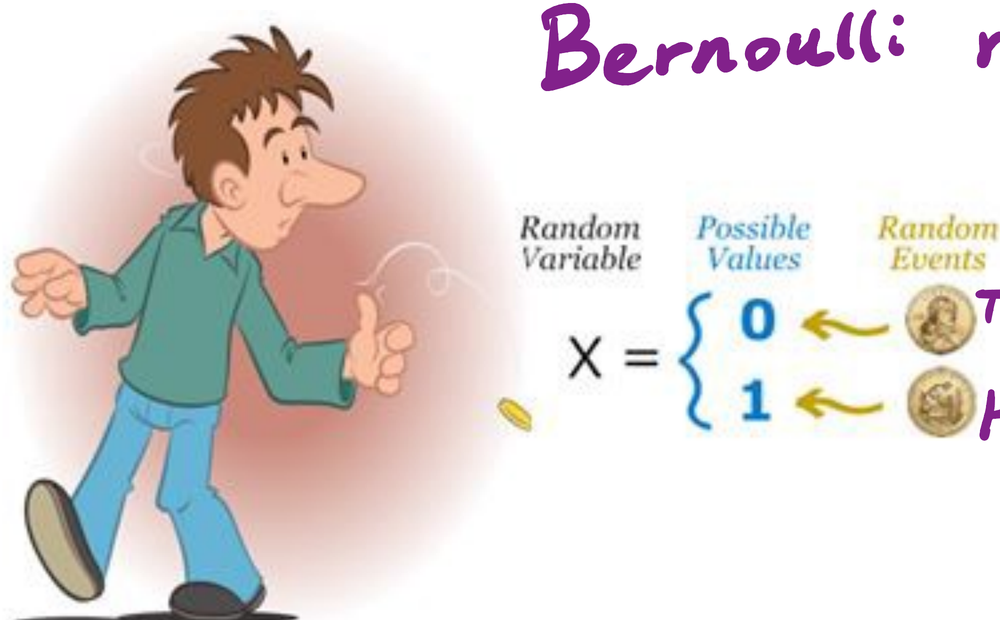
Random variables

A random variable maps all outcomes to Numbers,

$(\omega) \rightarrow (x)$

outcomes are disjoint

Bernoulli random variable



Tail

Head

$P(\omega = \text{head}) = p$

$P(\omega = \text{tail}) = 1-p$

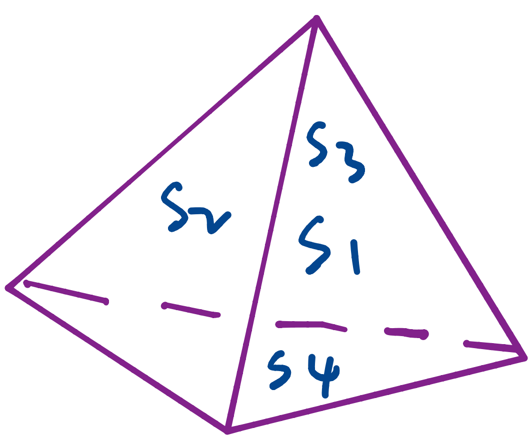
Random variables

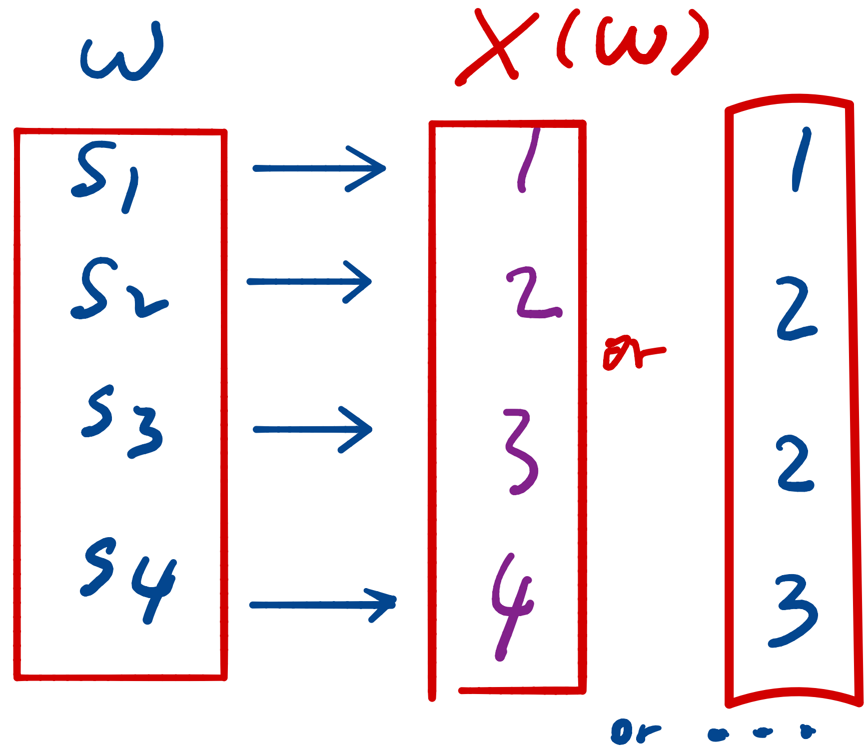
- The values of a random variable can be either **discrete**, **continuous** or **mixed**.

Discrete Random variables

- The range of a discrete random variable is a countable set of real numbers.

4-die

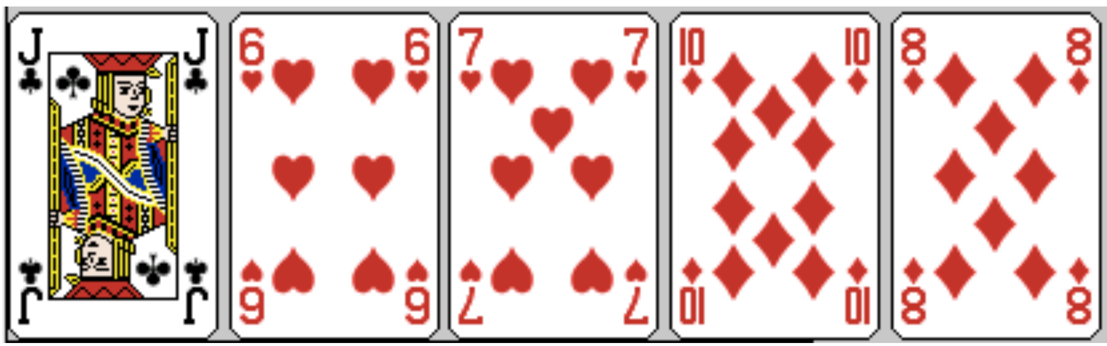




Random Variable Example

- **Number of pairs in a hand of 5 cards**

$\omega_i =$



$X(\omega_i) = \ ? \quad = 0$

- Let a single outcome be the hand of 5 cards
- Each outcome maps to values in the set of numbers {0, 1, 2}

0, 1, 2 are the possible values

Random Variable Example

- **Number of pairs in a hand of 6 cards**
  - Let a single outcome be the hand of 6 cards
  - What is the range of values of this random variable?

$X(\omega)$ could take

[0, 1, 2, 3]

Q: Random Variable

- If we roll a 3-sided fair die, and define random variable *U, such that*

$$U(\omega) = \begin{cases} -1 & \omega \to \text{side } 1 \\ 0 & \omega \to \text{side } 2 \\ 1 & \omega \to \text{side } 3 \end{cases} \qquad X = U^2$$

what is the range of X?

$$X(\omega) = \begin{cases} 0 & \omega \to \text{side } 2 \\ 1 & \omega \to \text{side } 1 \text{ or } 3 \end{cases}$$

A. {-1, 0, 1}

B. {0, 1}

Three important facts of Random variables

- Random variables have **probability functions**
- Random variables can be **conditioned** on events or other random variables
- Random variables have **averages**

Random variables have **probability functions**

- Let X be a random variable
- The set of outcomes $\{\omega_i \in \Omega, \text{ s.t. } X(\omega_i) = x_0\}$

  is an event with probability

$$P(X = x_0)$$

X is the random variable

x_0 is any unique instance that X takes on

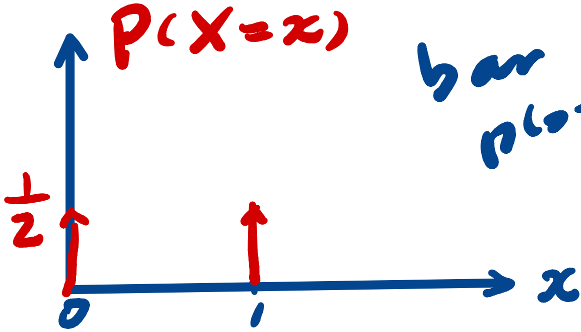
Probability Distribution

- $P(X = x)$ is called the probability distribution for all possible x
- $P(X = x)$ is also denoted as $P(x)$ or $p(x)$
- $P(X = x) \geq 0$ for all values that X can take, and is 0 everywhere else
- The sum of the probability distribution is 1 $$\sum_x P(x) = 1$$

All possible x are disjoint

Examples of Probability Distributions

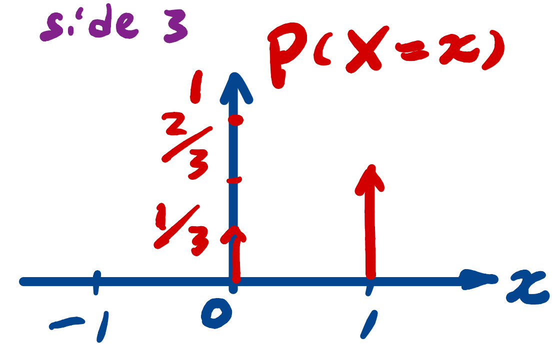
$$X(\omega) = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases} \quad \text{(fair coin)}$$



bar plot

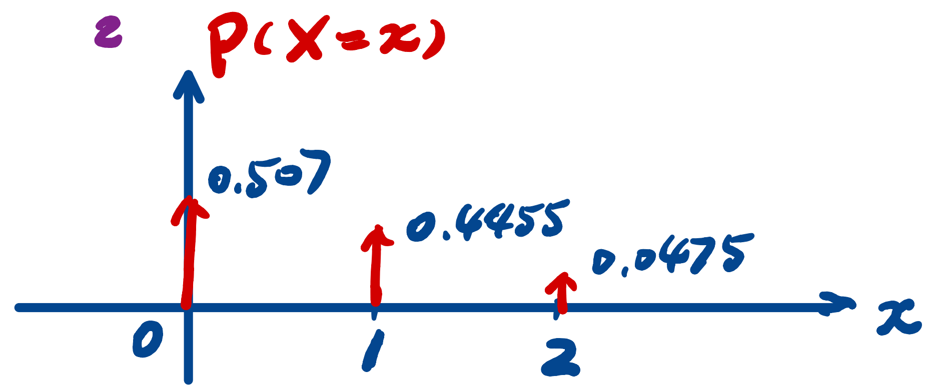
$$U(\omega) = \begin{cases} -1 & \text{side 1} \\ 0 & \text{side 2} \\ 1 & \text{side 3} \end{cases} \quad \text{fair die}$$

$$X(\omega) = U^2$$



$$X(\omega) = \begin{cases} 0 & \text{\# of pairs} = 0 \\ 1 & \phantom{\text{\# of pairs} = {}} 1 \\ 2 & \phantom{\text{\# of pairs} = {}} 2 \end{cases}$$

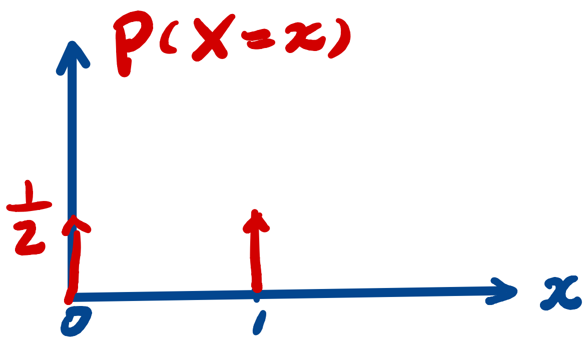
pairs in a hand of 5-cards



Another way to write PDF

$$X(\omega) = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$$

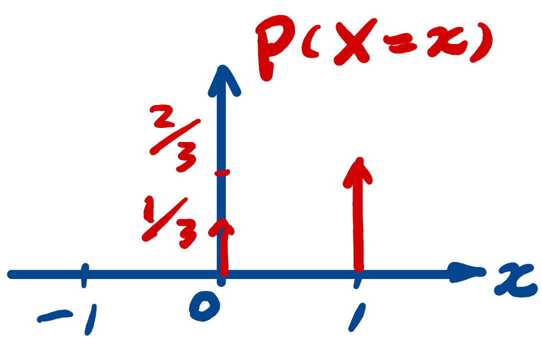
$$P(X=x) = \begin{cases} \frac{1}{2} & x=0 \\ \frac{1}{2} & x=1 \\ 0 & \text{otherwise} \end{cases}$$



$$U(\omega) = \begin{cases} -1 & \text{side } 1 \\ 0 & \text{side } 2 \\ 1 & \text{side } 3 \end{cases}$$

$$X(\omega) = U^2$$

$$P(X=x) = \begin{cases} \frac{1}{3} & x=0 \\ \frac{2}{3} & x=1 \\ 0 & \text{otherwise} \end{cases}$$



Cumulative distribution

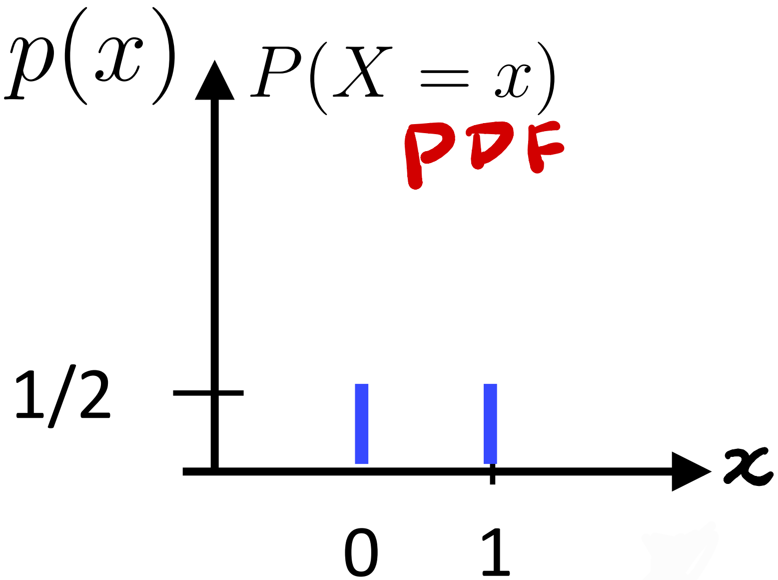
- $P(X \le x)$ is called the cumulative distribution function of X
- $P(X \le x)$ is also denoted as $f(x)$
- $P(X \le x)$ is a non-decreasing function of x

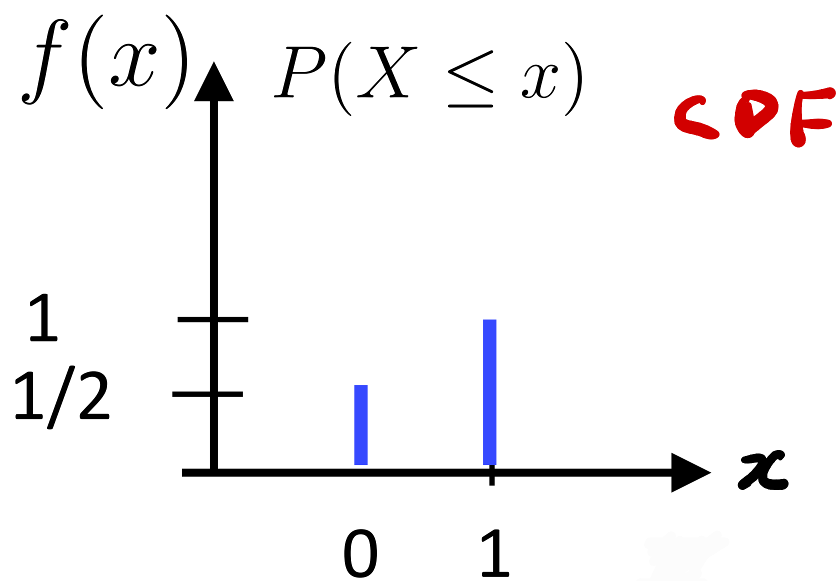
if $x =$ max of range of x
$P(X \le x) = 1$

Probability distribution and cumulative distribution

- Give the random variable ***X***,

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

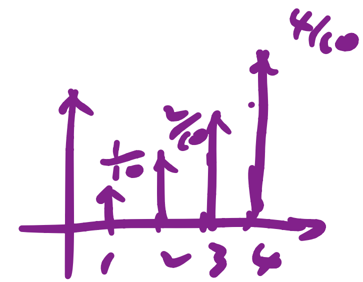




What is the value ?

A biased four-sided die is rolled once. Random variable X is defined to be the down-face value.

$$P(X=x)=\begin{cases}\frac{x}{10} & x=1,2,3,4 \\ 0 & \text{otherwise}\end{cases}$$



$P(X \le 4)$

A) 0.1

B) 0.3

C) 0.2

D) 0.6

$\boxed{\text{E)}}$ 1

Functions of Random Variables

$X = |U|$

$\max |U|$

$\ldots$

$X = U_1 + U_2$

$S = X + Y + \ldots$

$D = X - Y$

Q. Are these random variables the same?

$$X(w) = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases} \qquad Y(w) = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases}$$

same fair coin

$0 \to TT$

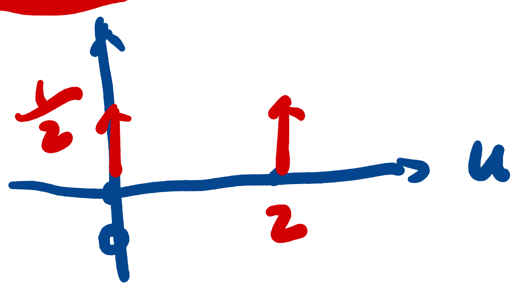
$U = 2X \qquad V = X + Y$

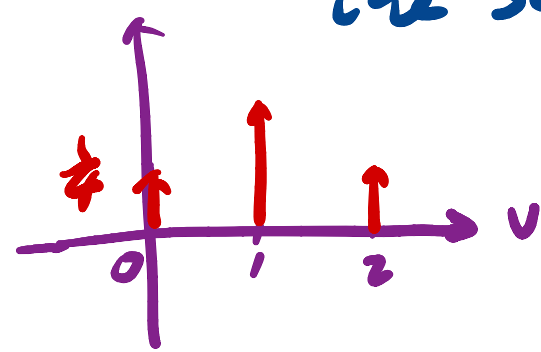
Are U and V the same?

A) Yes

$\boxed{\text{B)}}$ No.

whether pdfs are the same?





Function of random variables: die example

Roll 4-sided fair die twice. (randomly)

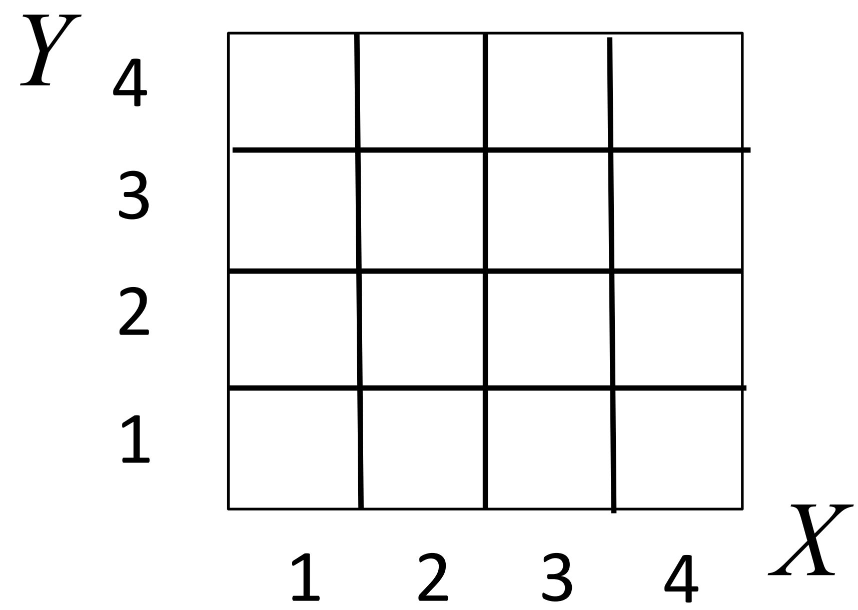
Define these random variables:

X, the values of 1^{st} roll

Y, the values of 2^{nd} roll

Sum $S = X + Y$

Difference $D = X - Y$



Size of Sample Space = ?

Random variable: die example

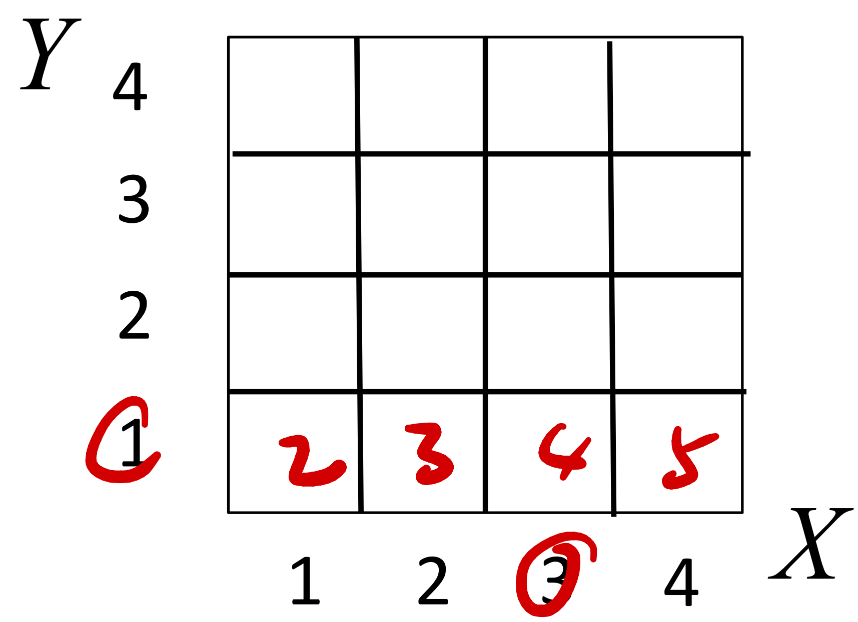
Roll 4-sided fair die twice.

$$P(X = 1)$$

$$P(Y \leq 2)$$

$$P(S = 7)$$

$$P(D \leq -1)$$



Size of Sample Space = 16

Random variable: die example

$S = X + Y$

Y \ X	1	2	3	4
4	5	6	7	8
3	4	5	6	7
2	3	4	5	6
1	2	3	4	5

$D = X - Y$

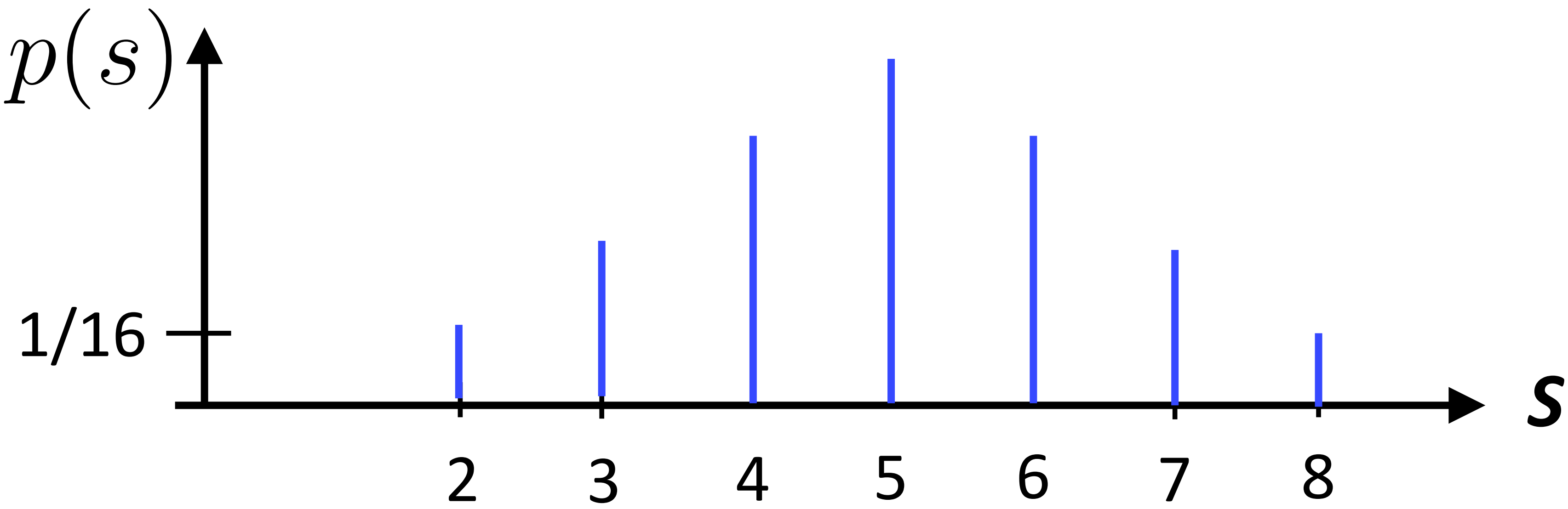
Y \ X	1	2	3	4
4	-3	-2	-1	0
3	-2	-1	0	1
2	-1	0	1	2
1	0	1	2	3

$P(S=7) = \frac{2}{16} = \frac{1}{8}$

$P(D \leq -1) = \frac{6}{16}$

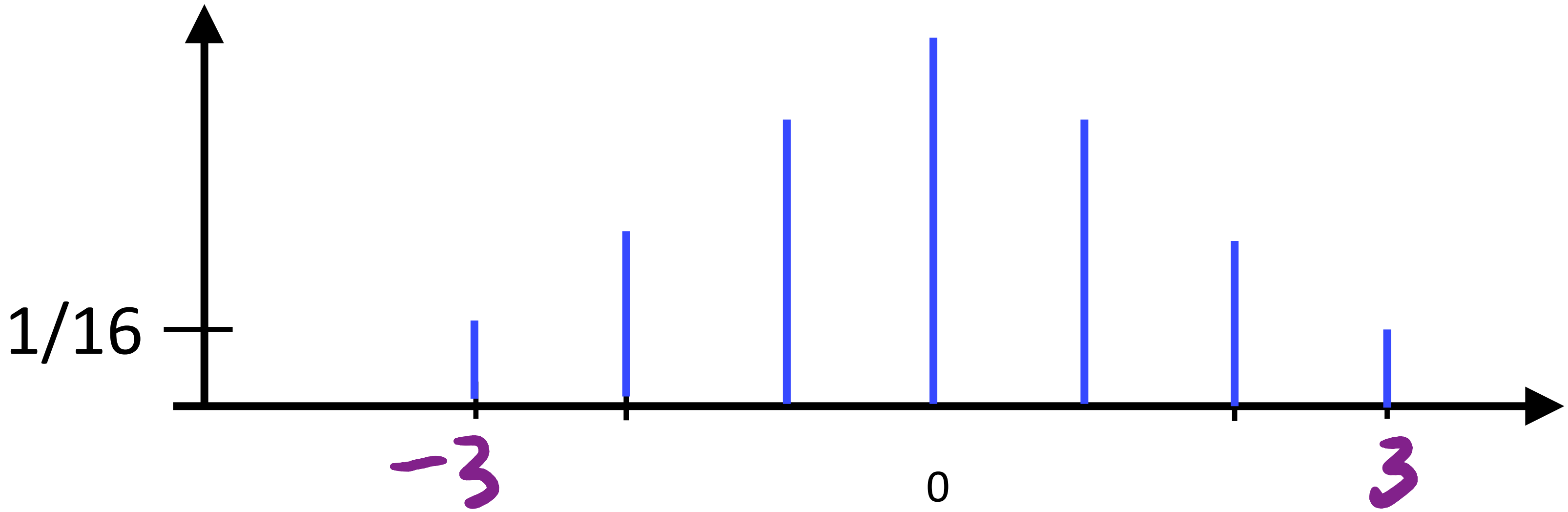
Probability distribution of the sum of two random variables

- Give the random variable ***S*** *in the 4-sided die, whose range is* {2,3,4,5,6,7,8}, probability distribution of ***S*.**



Probability distribution of the difference of two random variables

- Give the random variable ***D = X-Y***, *what is the probability distribution of **D**?*



Assignments

- Module Week 4, HW3 due Fri. night, quiz.
- Next time: More random variable, Expectations, Variance

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

See you next time

See You!