"A major use of probability in statistical inference is the updating of probabilities when certain events are observed" – Prof. M.H. DeGroot
Objectives

- Conditional Probability
- Review
- Total probability
- Independence
Conditional Probability

The probability of \( A \) given \( B \)

\[
P(A|B) = \frac{P(A \cap B)}{P(B)}
\]

\( P(B) \neq 0 \)

The line-crossed area is the new sample space for conditional \( P(A|B) \)
Joint Probability Calculation

\[ P(A \cap B) = P(A|B)P(B) \]

\[ P(soup \cap meat) = P(meat|soup)P(soup) = 0.5 \times 0.8 = 0.4 \]
Given the definition of conditional probability and the symmetry of joint probability, we have:

\[ P(A|B)P(B) = P(A \cap B) = P(B \cap A) = P(B|A)P(A) \]

And it leads to the famous Bayes rule:

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]
Total probability
Total probability general form
Total probability:
Bayes rule using total prob.
Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is 1/100,000. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is \( P(D|T) \), the probability of having disease given a positive test result?

\[
P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}
\]

Using total prob.
Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is \( \frac{1}{100,000} \). If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is \( P(D|T) \), the probability of having disease given a positive test result?

\[
P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}
\]
Independence

One definition:

\[ P(A|B) = P(A) \quad \text{or} \]
\[ P(B|A) = P(B) \]

Whether A happened doesn’t change the probability of B and vice versa.
Suppose that we have a fair coin and it is tossed twice. Let $A$ be the event “the first toss is a head” and $B$ the event “the two outcomes are the same.”

These two events are independent!
Independence

Alternative definition

LHS by definition

\[
\frac{P(A|B)}{P(B)} = \frac{P(A)}{P(A \cap B)} = P(A)
\]

\[
P(A \cap B) = P(A)P(B)
\]
Suppose you draw one card from a standard deck of cards. \( E_1 \) is the event that the card is a King, Queen or Jack. \( E_2 \) is the event the card is a Heart. Are \( E_1 \) and \( E_2 \) independent?
Pairwise independence is not mutual independence in larger context

\[
P(A_1) = P(A_2) = P(A_3) = P(A_4) = \frac{1}{4}
\]

\[
A = A_1 \cup A_2; P(A)
\]

\[
B = A_1 \cup A_3; P(B)
\]

\[
C = A_1 \cup A_4; P(C)
\]

\* \(P(ABC) \) is the shorthand for \( P(A \cap B \cap C) \)
Mutual independence

- Mutual independence of a collection of events $A_1, A_2, A_3...A_n$ is:

$$P(A_i | A_j A_k ... A_p) = P(A_i)$$

$$j, k, ... p \neq i$$

- It’s very strong independence!
An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that the flight is overbooked?
Probability using the property of Independence: Airline overbooking (1)

An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that the flight is overbooked?

$P(7 \text{ passengers showed up})$
An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that exactly 6 people showed up?

$P(6 \text{ people showed up}) = \binom{8}{6} p^6 (1-p)^2$
An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that the flight is overbooked?

$P(\text{overbooked}) =$
An airline has a flight with $s$ seats. They always sell $t$ ($t>s$) tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that exactly $u$ people showed up?

$$P(\text{exactly } u \text{ people showed up})$$
An airline has a flight with $s$ seats. They always sell $t$ ($t > s$) tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that the flight is overbooked?

$P(\text{overbooked})$
Q. Two disjoint events that have probability > 0 are certainly dependent to each other.

A. True

B. False
Q. Any event is independent of empty event B.

A. True

B. False
Assume event $A$ and $B$ are pairwise independent.

Given $C$, $A$ and $B$ are not independent any more because they become disjoint.
Event $A$ and $B$ are conditional independent given event $C$ if the following is true.

$$P(A \cap B | C) = P(A | C)P(B | C)$$

See an example in Degroot et al. Example 2.2.10
Assignments

- Module week3 on Compass
- Next time: Random variable
Additional References

- Charles M. Grinstead and J. Laurie Snell
  "Introduction to Probability"

- Morris H. Degroot and Mark J. Schervish
  "Probability and Statistics"
See you next time

See You!