Can we call $e$ the exciting $e$?

$$e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n$$

Credit: wikipedia

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Last time

- **Poisson distribution**
  \[ p(X = k) = \frac{e^{-\lambda} \lambda^k}{k!} \quad k \geq 0 \]

- Continuous random variable; uniform distribution

- **Exponential distribution**
Objectives

- Exponential distribution
- Normal (Gaussian) distribution
Exponential distribution

- Common Model for waiting time
- Associated with the Poisson distribution with the same $\lambda$

$p(x) = \lambda e^{-\lambda x}$ for $x \geq 0$

Credit: wikipedia
A continuous random variable $X$ is exponential if it represents the "time" until the next incident in a Poisson distribution with intensity $\lambda$. Proof:
See Degroot et al. Pg 324.

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0$$

It's similar to Geometric distribution – the discrete version of waiting in queue

Memoryless property
A continuous random variable $X$ is exponential if it represents the “time” until the next incident in a Poisson distribution with intensity $\lambda$.

$$p(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x \geq 0$$

$$E[X] = \frac{1}{\lambda} \quad \& \quad var[X] = \frac{1}{\lambda^2}$$
Example of exponential distribution

How long will it take until the next call to be received by a call center? Suppose it’s a random variable $T$. If the number of incoming call is a Poisson distribution with intensity $\lambda = 20$ in an hour. What is the expected time for $T$?

\[ E[T] = \frac{1}{\lambda} \text{ hr} \]

\[ = \frac{1}{20} = 3 \text{ min} \]
A store has a number of customers coming on Sat. that can be modeled as a Poisson distribution. In order to measure the average rate of customers in the day, the staff recorded the time between the arrival of customers, can he reach the same goal?

A. Yes  B. No

\[ E[T] = \frac{1}{\lambda} \]
Normal (Gaussian) distribution

The most famous continuous random variable distribution. The probability density is this:

\[
p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]

\[
\int_{-\infty}^{\infty} p(x) \, dx = 1
\]

Carl F. Gauss
(1777-1855)
Credit: wikipedia
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\[ E[X] = \mu \quad \& \quad var[X] = \sigma^2 \]

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Normal (Gaussian) distribution

🌟 The most famous continuous random variable distribution.

\[
p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]

\[
\int_{-\infty}^{+\infty} p(x) \, dx = 1
\]

\[
E[X] = \mu \quad \& \quad \text{var}[X] = \sigma^2
\]

Carl F. Gauss
(1777-1855)
Credit: wikipedia
A lot of data in nature are approximately normally distributed, ie. Adult height, etc.

\[ p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left( -\frac{(x - \mu)^2}{2\sigma^2} \right) \]

\[ E[X] = \mu \quad & \quad \text{var}[X] = \sigma^2 \]

Carl F. Gauss
(1777-1855)
Credit: wikipedia
PDF and CDF of normal distribution curves

PDF: \[ p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

CDF: \[ F(x) = \int_{-\infty}^{x} p(t) \, dt \]

Credit: wikipedia
Quantile

Quantiles give a measure of location, the median is the 0.5 quantile.

Credit: J. Orloff et al

\[ q_{0.6} : \text{left tail area} = 0.6 \iff F(q_{0.6}) = 0.6 \]
What is the value of 50% quantile in a standard normal distribution?

A. -1
B. 0
C. 1
Spread of normal (Gaussian) distributed data

99.7% of the data are within 3 standard deviations of the mean.

95% within 2 standard deviations.

68% within 1 standard deviation.

Credit: wikipedia
What is this probability?

\[ X \sim N(\mu, \sigma^2) \]

\[ p(a < x < b) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{a}^{b} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \]

No analytical solution!
If we standardize the normal distribution (by subtracting $\mu$ and dividing by $\sigma$), we get a random variable that has standard normal distribution.

A continuous random variable $X$ is **standard normal** if

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$
Derivation of standard normal distribution

\[
\begin{align*}
& \int_{-\infty}^{+\infty} p(x) \, dx \\
= & \int_{-\infty}^{+\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp\left( - \frac{(x - \mu)^2}{2\sigma^2} \right) \, dx \\
= & \int_{-\infty}^{+\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp\left( - \frac{\hat{x}^2}{2} \right) \sigma \, d\hat{x} \\
= & \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left( - \frac{\hat{x}^2}{2} \right) \, d\hat{x} \\
= & \int_{-\infty}^{+\infty} p(\hat{x}) \, d\hat{x}
\end{align*}
\]

Call this standard and omit using a hat

\[ p(x) = \frac{1}{\sqrt{2\pi}} \exp\left( - \frac{x^2}{2} \right) \]
What is this probability?

\[ p(a < x < b) = \left(\frac{x - \mu}{\sigma}\right)^2 \]

\[ = \int_{a}^{b} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \, dx = \int_{\frac{b-\mu}{\sigma}}^{\frac{-c_2}{\sqrt{2\pi}}} e^{-\frac{\hat{x}^2}{2}} \, d\hat{x} \]
Q. What is the mean of standard normal?

A. 0
B. 1
Q. What is the standard deviation of standard normal?

A. 0
B. 1
Standard normal distribution

- If we standardize the normal distribution (by subtracting \( \mu \) and dividing by \( \sigma \)), we get a random variable that has standard normal distribution.

- A continuous random variable \( X \) is **standard normal** if

\[
p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)
\]

\[
E[X] = 0 \quad \& \quad \text{var}[X] = 1
\]
Another way to check the spread of normal distributed data

- Fraction of normal data within 1 standard deviation from the mean.

\[
\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{x^2}{2}} dx \simeq 0.68
\]

- Fraction of normal data within \( k \) standard deviations from the mean.

\[
\frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-\frac{x^2}{2}} dx
\]
Using the standard normal’s table to calculate for a normal distribution’s probability

If \( X \sim N (\mu=3, \sigma^2 =16) \) (normal distribution)

\[
P(X \leq 5) = \text{?}
\]

\[
P(\frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}) = 0.6915
\]

\[
\begin{array}{cccccccccccc}
& .00 & .01 & .02 & .03 & .04 & .05 & .06 & .07 & .08 & .09 \\
0.0 & .0000 & .0040 & .0080 & .0120 & .0160 & .0199 & .0239 & .0279 & .0319 & .0359 \\
0.1 & .0398 & .0438 & .0478 & .0517 & .0557 & .0596 & .0636 & .0675 & .0714 & .0753 \\
0.2 & .0573 & .0632 & .0682 & .0732 & .0782 & .0832 & .0882 & .0932 & .0982 & .1032 \\
0.3 & .0617 & .0659 & .0700 & .0740 & .0781 & .0821 & .0862 & .0902 & .0943 & .0984 \\
0.4 & .0655 & .0692 & .0728 & .0765 & .0802 & .0839 & .0876 & .0913 & .0950 & .0987 \\
0.5 & .0691 & .0722 & .0754 & .0785 & .0817 & .0849 & .0882 & .0914 & .0946 & .0978 \\
0.6 & .0726 & .0758 & .0790 & .0823 & .0856 & .0889 & .0922 & .0955 & .0988 & .1021 \\
0.7 & .0758 & .0791 & .0824 & .0858 & .0892 & .0926 & .0960 & .0994 & .1028 & .1062 \\
0.8 & .0788 & .0821 & .0855 & .0889 & .0924 & .0958 & .0993 & .1027 & .1062 & .1096 \\
0.9 & .0818 & .0852 & .0887 & .0922 & .0957 & .0992 & .1027 & .1062 & .1097 & .1132 \\
1.1 & .0880 & .0916 & .0952 & .0987 & .1023 & .1058 & .1094 & .1129 & .1164 & .1200 \\
\end{array}
\]

CDF table of \( N(0,1) \)

\[
\int_{-\infty}^{x} p(x) \, dx
\]

\( \mu_X = \frac{X_0}{2} = 0.5 \)

\( \sigma_X = 0.5 \)
Q. Is the table with only positive x values enough?

A. Yes  B. No.

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Central limit theorem (CLT)

- The distribution of the sum of $N$ independent identical (IID) random variables tends toward a normal distribution as $N \rightarrow \infty$

- Even when the component random variables are not exactly IID, the result is approximately true and very useful in practice
Central limit theorem (CLT)

- CLT helps explain the prevalence of normal distributions in nature.
- A binomial random variable tends toward a normal distribution when $N$ is large due to the fact it is the sum of IID Bernoulli random variables.

Mathematically:

$$X_{bin} = \sum_{i=1}^{N} X_i$$

- $X_i \sim \{0, 1\}$
- $p(x_i) = \begin{cases} p & x_i = 1 \\ 1-p & x_i = 0 \end{cases}$
The Binomial distribution looks very similar to Normal when N is large.

\[
P(X_{\text{Bin}} = k) = \binom{N}{k} p^k (1-p)^{N-k}
\]
Binomial approximation with Normal

**Binomial distribution**

\[ E[X_{B:n}] = np \]
\[ \text{Var}(X_{B:n}) = np(1-p) \]

**Approximation with Normal**

\[ \mu = 20, \quad \sigma^2 = 10 \]
Binomial approximation with Normal

- Let $k$ be the number of heads appeared in 40 tosses of fair coin

- The goal is to estimate the following with normal

$$P(10 \leq k \leq 25) = \sum_{k=10}^{25} \binom{40}{k} 0.5^k 0.5^{40-k}$$

$$= \sum_{k=10}^{25} \left( \frac{40!}{k!(40-k)!} \right) 0.5^{40} \approx 0.96$$

$$E[k] = np = 40 \cdot 0.5 = 20$$

$$std[k] = \sqrt{np(1-p)} = \sqrt{40 \cdot 0.5 \cdot 0.5} = \sqrt{10}$$
Binomial approximation with Normal

- Use the same mean and standard deviation of the original binomial distribution.
  \[ \mu = 20 \quad \sigma = \sqrt{10} \approx 3.16 \]

- Then standardize the normal to do the calculation
  \[
P(10 \leq k \leq 25) \approx \frac{1}{\sigma \sqrt{2\pi}} \int_{10}^{25} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx
  \]
  \[
  = \frac{1}{\sqrt{2\pi}} \int_{\frac{10-20}{3.16}}^{\frac{25-20}{3.16}} \exp\left(-\frac{x^2}{2}\right) dx
  \]
  \[
  \approx 0.94
  \]
Assignments this week

- Week 6 quiz
- HW5
- Prepare for Midterm 1:
  - Practice exams
  - Read through instructions on Compass
Additional References

- Charles M. Grinstead and J. Laurie Snell
  "Introduction to Probability”

- Morris H. Degroot and Mark J. Schervish
  "Probability and Statistics”
See you next time

See You!