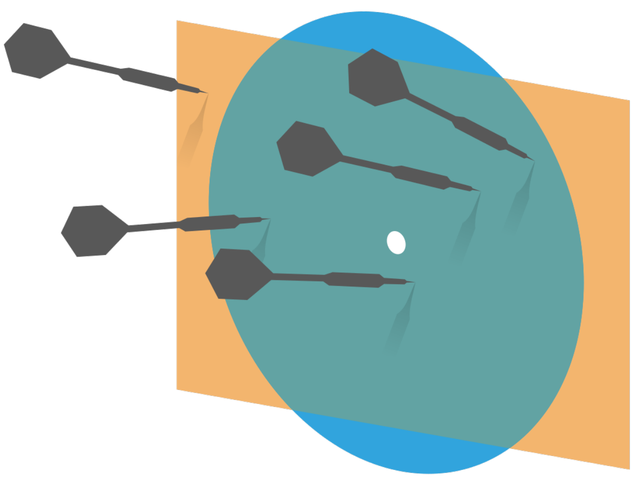
Probability and Statistics for Computer Science



Credit: wikipedia

"Probabilistic analysis is mathematical, but intuition dominates and guides the math" – Prof. Dimitri Bertsekas

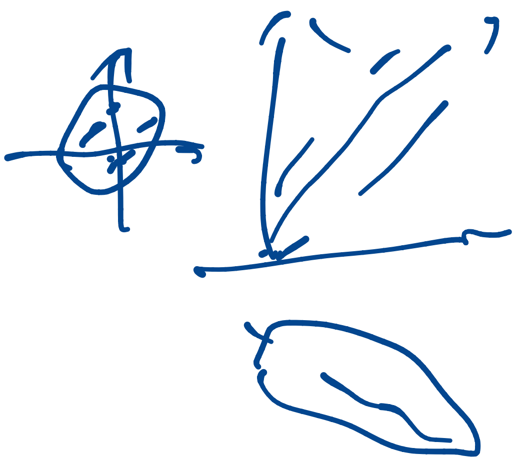
Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 9.3.2020

Homework (I)

- Due 9/3 today at 11:59pm
- There is one optional problem with extra 5 points. (Won't be in exams)

Last time

- Median, Interquartile range, box plot and outlier , Mode & Skew
- Scatter plots, Correlation Coefficient



$[-1, 1]$

$$\frac{\sum \hat{x}\hat{y}}{N}$$

Objectives

Probability: a first look

Definitions

Random Experiment.

Outcome, Sample Space, Event

probability – three axioms

Properties of probability

Calculating probability

A game of chance

http://www.randomservices.org/random/apps/RouletteExperiment.html

Q. Will I win?

A. Yes B. No

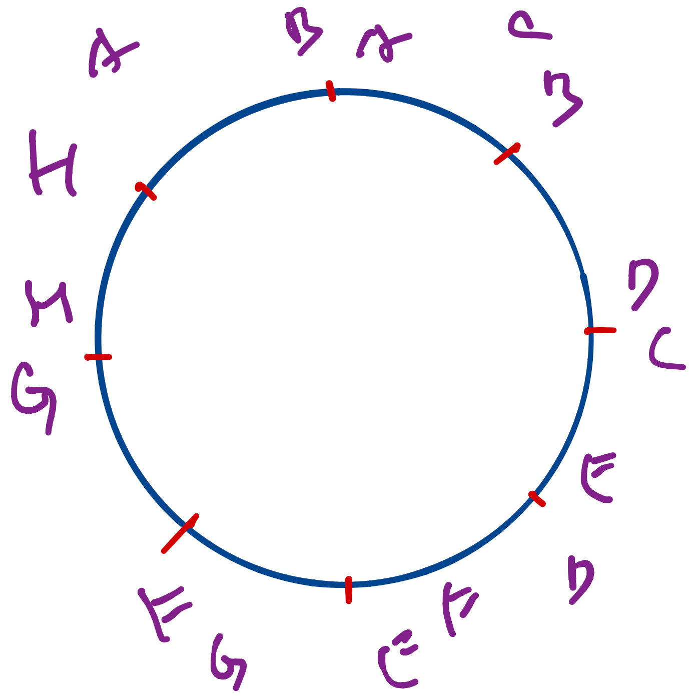
Def. of Random experiment

Random

Repeatable

How many arrangements are there?

Arrange 8 people to sit by a round table.



$$7!$$

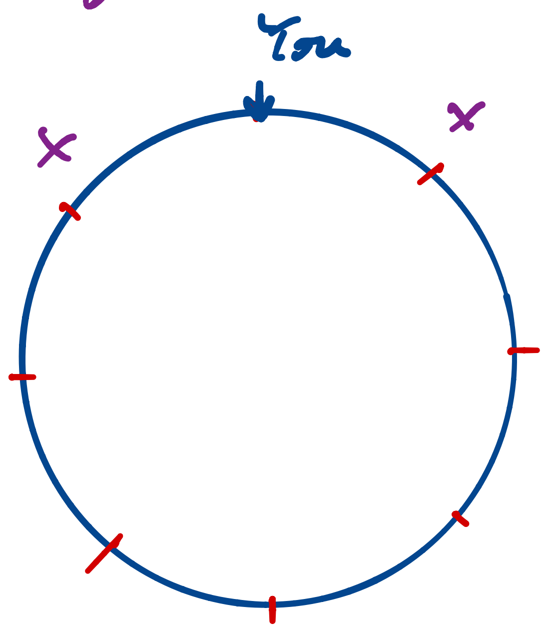
$$\frac{8!}{8} = 7!$$

8 Redund..

How much is the chance?

You & your best friend

sit together. (Seats are randomly assigned)



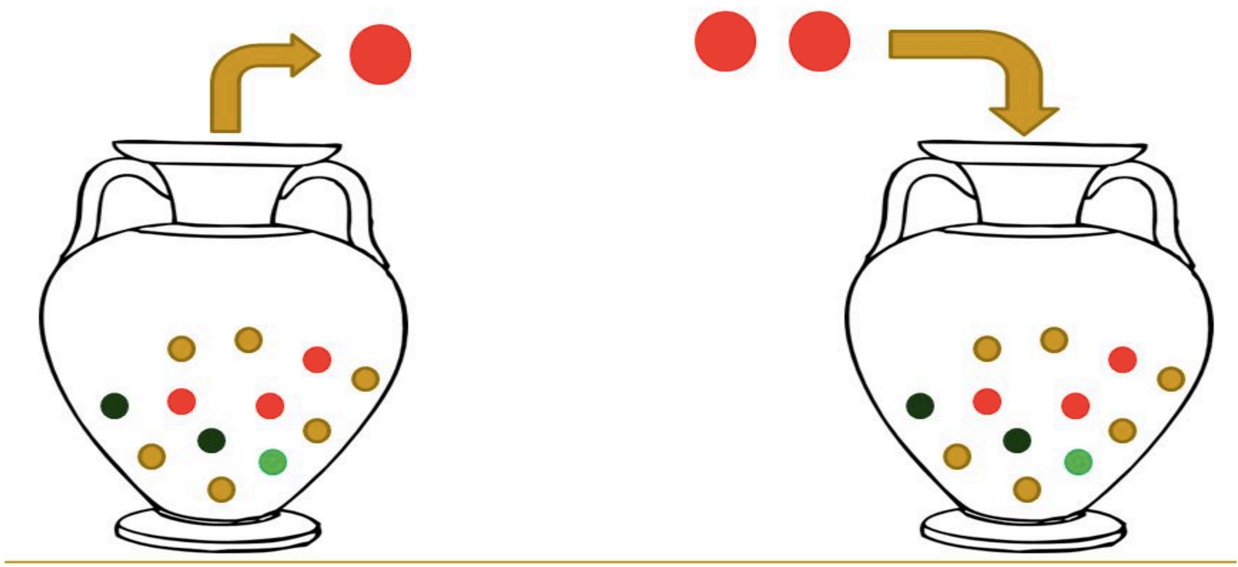
$$2 \times 6!$$

$$\text{probability} = \frac{2 \times 6!}{7!} = \frac{2}{7}$$

Outcome

- An outcome **A** is a possible result of a random repeatable experiment

Random: uncertain, Nondeter-ministic, …



Sample space

- The Sample Space, **Ω,** is the set of all possible outcomes associated with the random experiment
- Discrete or Continuous

Sample Space example (1)

- Experiment: we roll a tetrahedral die twice

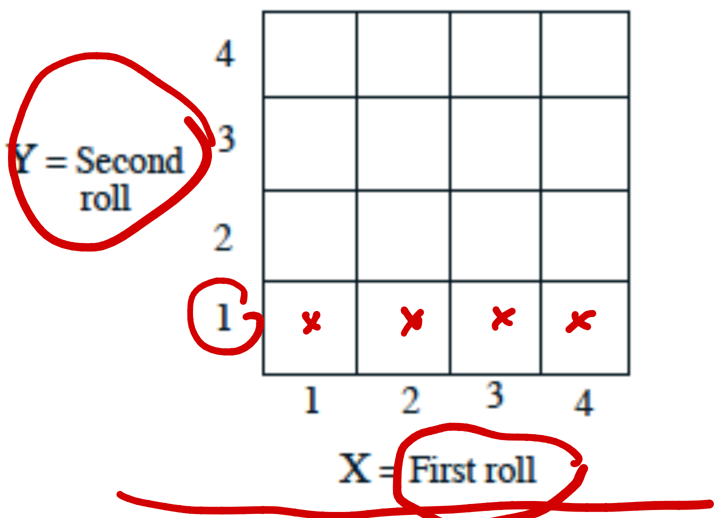
1 2 3 4 16

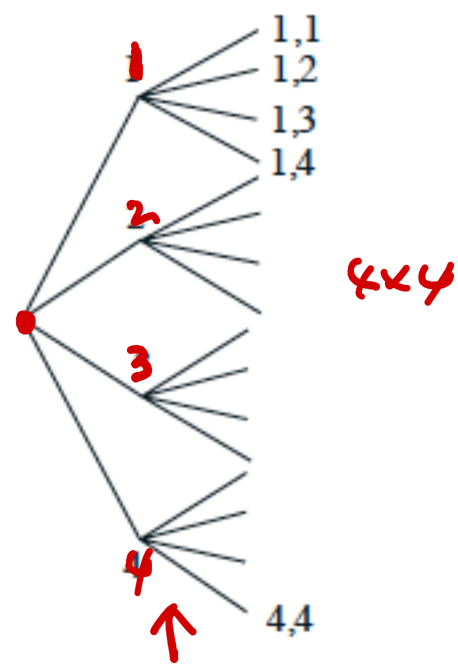
1 2 3 4

- **Discrete** Sample space:

1st 2nd 1st 2nd

{(1,1), (1,2)....}

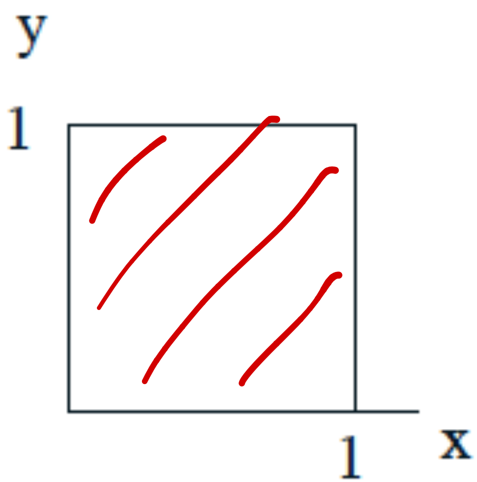




Sample Space example (2)

- Experiment: Romeo and Juliet's date
- **Continuous** Sample space:

$$\Omega = \{(x, y) \mid\ 0 \leq x, y \leq 1\}$$



Sample Space depends on experiment (3)

- Different coin tosses
  - Toss a fair coin H T
  - Toss a fair coin twice 4

    HH
    HT
    TH
    TT
  - Toss until a head appears

    H TH TTH . . .

Sample Space depends on experiment (4)

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **with replacement**?

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **without replacement**?

Q.

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **with replacement**? What is the size of the sample space?

  A. 5 B. 7 C. 9

$3 \times 3 = 9$

Q.

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **without replacement**? What is the size of the sample space?

A. 5 B. 6 C. 9

$3 \times 2 = 6$

Sample Space in real life

- Possible outages of a power network
- Possible mutations in a gene

disc.

- A bus' arriving time

cont.

Event

- An event E is a subset of the sample space Ω
- So an event is a set of outcomes that is a subset of Ω, ie.
  - Zero outcome $\phi = \{ \}$
  - One outcome $\{A_1\}$
  - Several outcomes $\{A_1, A_2, A_3\}$
  - All outcomes Ω

The same experiment may have different events

- When two coins are tossed
  - Both coins come up the same?
  - At least one head comes up?

HH
TT

HT
HH
TH

Some experiment may never end

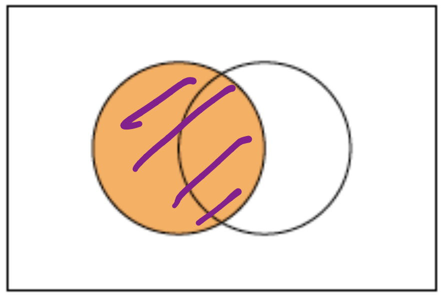
- Experiment: Tossing a coin until a head appears

  - E: Coin is tossed at least 3 times

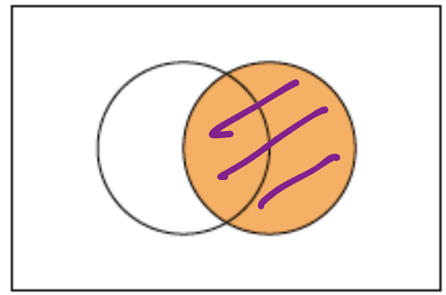
    This event includes infinite # of outcomes

Venn Diagrams of events as sets

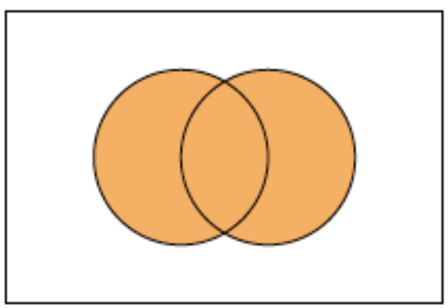
Ω



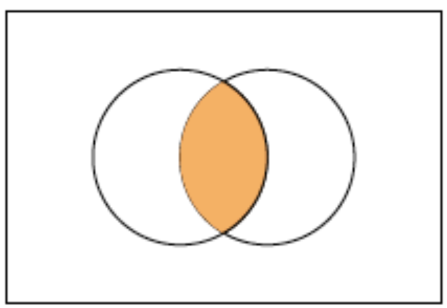
E_1



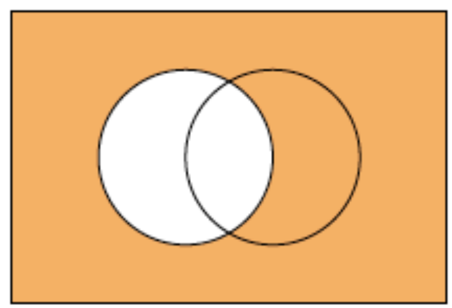
E_2



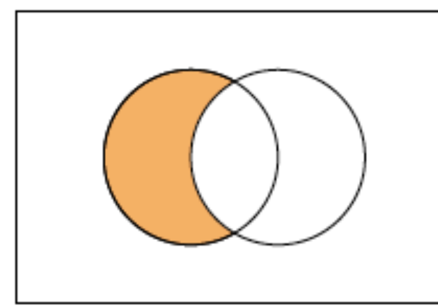
$E_1 \cup E_2$



$E_1 \cap E_2$



${E_1}^c$



$E_1 - E_2$

Combining events

- Say we roll a six-sided die. Let $$E_1 = \{1, 2, 5\} \; and \; E_2 = \{2, 4, 6\}$$
  - What is $E_1 \cup E_2$ — $\{1, 2, 4, 5, 6\}$
  - What is $E_1 \cap E_2$ — $\{2\}$
  - What is $E_1 - E_2$ — $\{1, 5\}$
  - What is $E_1^c = \Omega - E_1$ — $\Omega = \{1, 2, 3, 4, 5, 6\}$

$E_1^c = \{3, 4, 6\}$

Frequency Interpretation of Probability

- Given an experiment with an outcome **A**, we can calculate the probability of **A** by repeating the experiment over and over

$$P(A) = \lim_{N->\infty} \frac{number\ of\ time\ A\ occurs}{N}$$

- So,

$$0 \leq P(A) \leq 1$$

$$\sum_{A_i \in \Omega} P(A_i) = 1$$

Axiomatic Definition of Probability

- A probability function is any function P that maps sets to real number and satisfies the following **three** axioms:

  1) Probability of any event E is non-negative

$$P(E) \geq 0$$

E → number

  2) Every experiment has an outcome

$$P(\Omega) = 1$$

$\Omega \to 1$

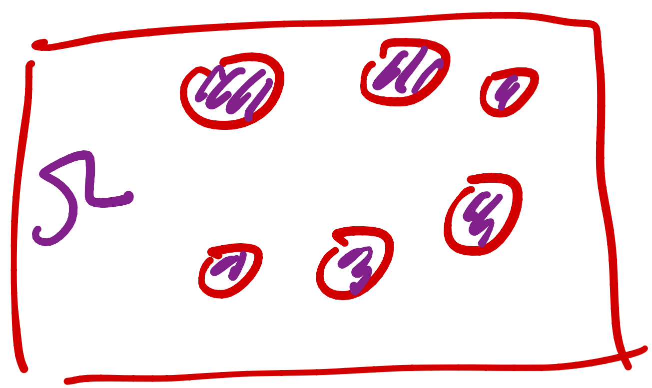
Axiomatic Definition of Probability

Mutually Exclusive

3) The probability of disjoint events is additive

$$P(E_1 \cup E_2 \cup \ldots \cup E_N) = \sum_{i=1}^{N} P(E_i)$$

$$if\ E_i \cap E_j = \emptyset\ for\ all\ i \neq j$$



Q.

- Toss a coin 3 times

The event "exactly 2 heads appears" and "exactly 2 tails appears" are disjoint.

A. True

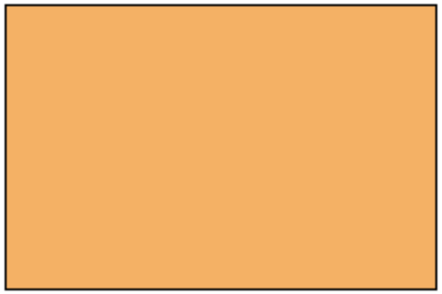
B. False

$2^3 = 8$

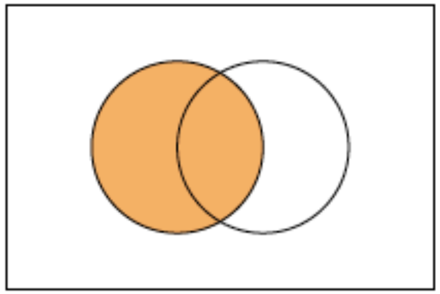
THH HHT

HTT TTH

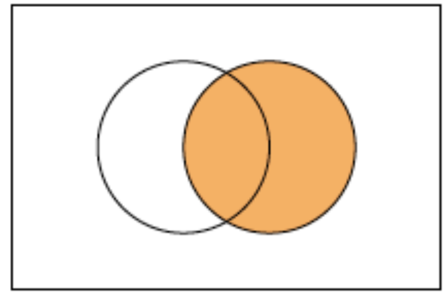
Venn Diagrams of events as sets



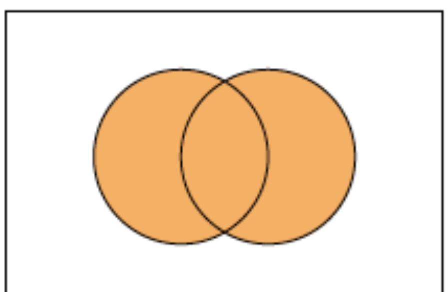
Ω



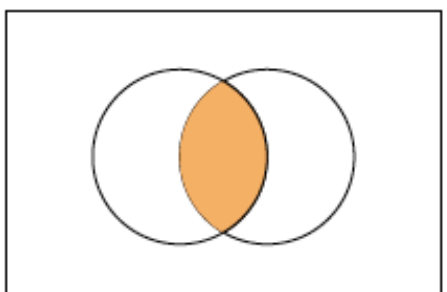
E_1



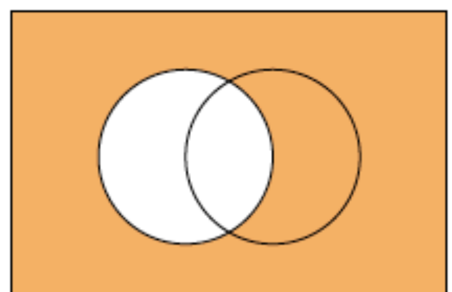
E_2



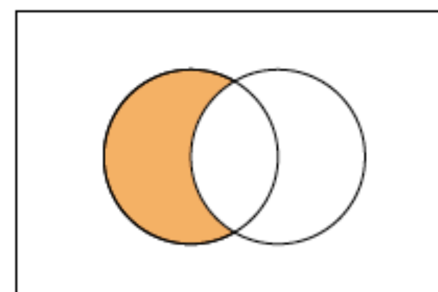
$E_1 \cup E_2$



$E_1 \cap E_2$



${E_1}^c$



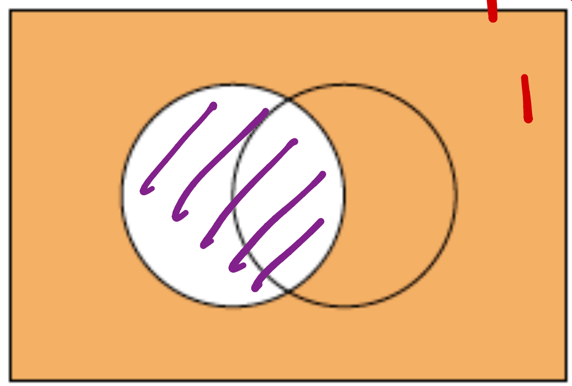
$E_1 - E_2$

Properties of probability

- The complement

$$P(E^c) = 1 - P(E)$$

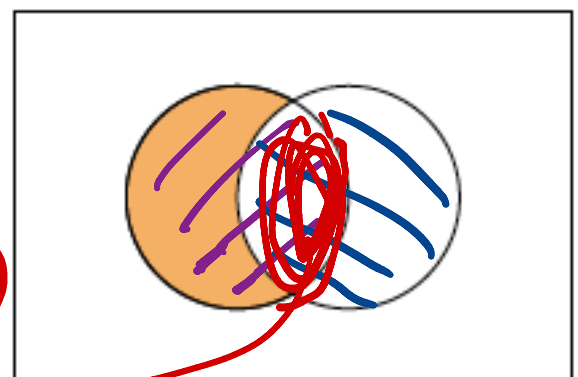
$P(\Omega) = 1$



$1 - P(E)$

- The difference

$$P(E_1 - E_2) = P(E_1) - P(E_1 \cap E_2)$$

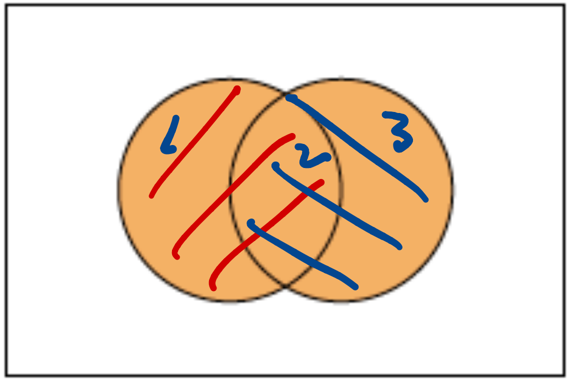


$P(E_1) - P(E_1 \cap E_2)$

Properties of probability

- The union

$$P(E_1 \cup E_2) =$$
$$P(E_1) + P(E_2)$$
$$- P(E_1 \cap E_2)$$



$P(E_1 \cup E_2) = P(1) + P(2) + P(3)$

- The union of multiple E

$P(E_1) + P(E_2)$
$= P(1) + P(2)$
$+ P(2) + P(3)$

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$$
$$- P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1)$$
$$+ P(E_1 \cap E_2 \cap E_3)$$

The Calculation of Probability

- Discrete countable finite event
- Discrete countable infinite event
- Continuous event

Counting to determine probability of countable finite event

- From the last axiom, the probability of event **E** is the sum of probabilities of the disjoint outcomes

$$P(E) = \sum_{A_i \in E} P(A_i)$$

A_i

- If the outcomes are atomic and have equal probability,

$$P(E) = \frac{number\ of\ outcomes\ in\ E}{total\ number\ of\ outcomes\ in\ \Omega}$$

Probability using counting: (1)

- Tossing a fair coin twice:
  - Prob. that it appears the same?

    $E = \{HH, TT\}$ $\quad 1/2$

    $\Omega = \{HH, HT, TH, TT\}$
  - Prob. that at least one head appears?

    $3/4$

Probability using counting: (2)

- 4 rolls of a 5-sided die:

  E: they all give different numbers

  - Number of outcomes that make the event happen:

    $5 \times 4 \times 3 \times 2 =$

  - Number of outcomes in the sample space

    $5 \times 5 \times 5 \times 5 = 5^4$

  - Probability:

    $\frac{5!}{5^4}$

Probability using counting: (2)

- What about N-1 rolls of a N-sided die?

  E: they all give different numbers

  - Number of outcomes that make the event happen:

    $N \cdot (N-1) \cdots \times 2$

  - Number of outcomes in the sample space

    N^N

  - Probability: $\frac{N!}{N^N}$

Probability by reasoning with the complement property

- If $P(E^c)$ is easier to calculate

$$P(E) = 1 - P(E^c)$$

Probability by reasoning with the complement property

- A person is taking a test with **N** true or false questions, and the chance he/she answers any question right is 50%, what's probability the person answers **at least** one question right?

E^c: none is right

$1, 2 \ldots N$

$\frac{1}{2} \times \frac{1}{2} \times \ldots \frac{1}{2}$

$P(E) = 1 - P(E^c) = 1 - \left(\frac{1}{2}\right)^N$

$N \to \infty$

$P(E) \to 1$

Probability by reasoning with the union property

- If E is either E1 or E2

$$P(E) = P(E_1 \cup E_2) =$$

$$P(E_1) + P(E_2) - \boxed{P(E_1 \cap E_2)}$$

Probability by reasoning with the properties (2)

- A person may ride a bike on any day of the year equally. What's the probability that he/she rides on a Sunday or on 15th of a month?

366 days this year

E_1 = Sunday

E_2: 15th of a month

2: 15th of month & Sun.

52 Sundays

$$P(E) = P(E_1 \cup E_2)$$
$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$
$$= \frac{52}{366} + \frac{12}{366} - \frac{2}{366}$$

Counting may not work

- This is one important reason to use the method of reasoning with properties

What if the event has outcomes

- Tossing a Fair coin until head appears
  - Coin is tossed at least 3 times

    This event includes infinite # of outcomes.

    And the outcomes don't have equal probability.

$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$ $\quad (\frac{1}{2})^4 \quad (\frac{1}{2})^5$

$\underline{\text{TTH}}$, $\underline{\text{TTTH}}$, $\underline{\text{TTTTH}}$....

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

See you next time

See You!