Discussion Problem

CS 173: Discrete Structures

Problem 1. Logical Reasoning

Suppose we are given the following facts:

1. All chestnut-eating animals are fun-loving
2. No penguin eats mulberries
3. Some well-dressed animals are uncomfortable
4. At least one penguin is uncomfortable
5. All animals eat mulberries or chestnuts
6. No uncomfortable animal eats mulberries

Which of the following statements can be proved (inferred) from the above facts?

You may assume that "not comfortable" is the same as "uncomfortable". You may also assume that penguins are known to be animals.

Select one or more:

(a) Every comfortable penguin eats mulberries

(b) At least one penguin is well-dressed

(c) No penguins are fun-loving

(d) There is at least one fun-loving well-dressed animal

(e) All penguins are fun-loving

(f) All penguins are uncomfortable

Solution:

First, we will model the given statements using (first-order) logic. *(This translation isn't strictly necessary, but in many cases it may be easier to see what's going on, and the concise statements may be easier to work with.)* We'll use the following predicates, where variables range over animals:

- $c(x)$: x is chestnut-eating
- $fl(x)$: x is fun-loving
- $penguin(x)$: x is a penguin
- $m(x)$: x eats mulberries
- $wd(x)$: x is well-dressed
- $comfy(x)$: x is comfortable

We can then formulate the facts as follows. Let us write them using *implications* as much as possible.

1. $\forall x.\ c(x) \Rightarrow fl(x)$
2. $\forall x.\ penguin(x) \Rightarrow (\neg m(x))$
3. $\exists x.\ wd(x) \wedge \neg comfy(x)$
4. $\exists x.\ penguin(x) \wedge \neg comfy(x)$
5. $\forall x.\ m(x) \vee c(x)$, which is equivalent to $\forall x.\ (\neg m(x)) \Rightarrow c(x)$ *(Note: this is* not *the same as* $\forall x.m(x) \vee \forall x.c(x)$*)*
6. $\neg\exists x.\ (\neg comfy(x) \wedge m(x))$, which is equivalent to $\forall x.\ (comfy(x) \vee \neg m(x))$, which is equivalent to $\forall x.\ (m(x) \Rightarrow comfy(x))$.

To aid reasoning, let's draw a graph that captures *universal* implications (we will leave the existential formulas out). In this graph, for every formula of the form $\forall x.\alpha(x) \Rightarrow \beta(x)$, we will throw in an arrow from $\alpha(x)$ to $\beta(x)$. Let us also throw in the equivalent contrapositives as well—since $\forall x.\alpha(x) \Rightarrow \beta(x)$ is equivalent to $\forall x.(\neg(\beta(x))) \Rightarrow (\neg(\alpha(x)))$, we will throw in an arrow from $\neg\beta(x)$ to $\neg\alpha(x)$ as well.

$$
\begin{tikzcd}
penguin(x) \arrow[r, "(2)"] & \neg m(x) \arrow[r, "(5)"] & c(x) \arrow[r, "(1)"] & fl(x) \\
\neg Comfy(x) \arrow[ur, "(6)"'] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\neg fl(x) \arrow[r, "(1)"] & \neg c(x) \arrow[r, "(5)"] & m(x) \arrow[r, "(2)"] \arrow[dr, "(6)"'] & \neg penguin(x) \\
& & & comfy(x)
\end{tikzcd}
$$

Note that the above captures only the universal statements, i.e, (1), (2), (5), and (6). It does not capture the existential statements (3) and (4).

Let us now turn to the various statements.

(a) Every comfortable penguin eats mulberries.

(It is tempting to say simply that the statement must be false because it directly contradicts Fact 2. However, notice that if the world contains zero comfortable penguins, this statement would actually be true.)

Let us construct a concrete counterexample to the statement. We'll start with a comfortable penguin called p that does *not* eat mulberries. Let's see if we can build a world with p that is consistent with all known facts.

(The counterexample you come up with does not have to be the same as ours - some of the choices we make below are forced by the facts, but others are up to you.)

(2) says no penguin eats mulberries. This is fine for p as it does not eat mulberries. (5) is also consistent, but in order to satisfy (5), we must allow that p eats chestnuts. (6) is consistent with our assumptions, since p is anyway comfortable, and (6) talks about uncomfortable animals only. (1) talks about chestnut-eating animals, and so we must allow that p is fun-loving. To be consistent with (4), we'll have to add a second penguin, p', which is uncomfortable. p' does not eat mulberries, eats chestnuts, is fun-loving (to satisfy (4)), and is not well-dressed. Finally, to be consistent with (3), we'll declare that p is not well-dressed, but then we need to add one more animal a. a is not a penguin, a is uncomfortable (to satisfy (3)). We can have this animal not eat mulberry (to satisfy (6)), and eat chestnuts (to satisfy (5)), and be fun-loving (to satisfy (1)).

So overall, our world W has exactly three animals, p, p', and a, which can be described as follows:

- $penguin(p)$, $comfy(p)$, $\neg m(p)$, $c(p)$, $fl(p)$, $\neg wd(p)$.
- $\neg penguin(a)$, $\neg comfy(a)$, $\neg m(a)$, $c(a)$, $fl(a)$, $wd(a)$.
- $penguin(p')$, $\neg comfy(p')$, $\neg m(p')$, $c(p')$, $fl(p')$, $\neg wd(p')$.

We see that this world satisfies all 6 facts but does not satisfy the given statement (which is, formally, $\forall x.(penguin(x) \wedge comfy(x)) \Rightarrow m(x))$.

Hence the given statement does not logically follow from the facts.

(b) At least one penguin is well-dressed.

The world W we constructed for the part (a) shows that there need not be a well-dressed penguin. So this statement is not entailed by the facts.

(The only fact that requires a penguin to exist at all is (4). That penguin is uncomfortable, but it needn't be well-dressed.)

(c) No penguins are fun-loving.

Again, the world W we painted above shows that there can be fun-loving penguins. So the statement is not entailed by the facts.

(d) There is at least one fun-loving well-dressed animal.

This is provable. We know from (3) that there is a well-dressed animal that is also uncomfortable. From the graph, we see that uncomfortable animals must be fun-loving.

A more formal argument (following the implications in the graph) is as follows:

- From (3), we know there is at least one well-dressed animal that is also uncomfortable; call this animal a.
- From (6), we know that a does not eat mulberries.
- From (5), we know that a must eat chestnuts.
- From (1), we know that a must be fun-loving.
- Hence there is at least one fun-loving well-dressed animal.

(e) All penguins are fun-loving.

From the graph, it's clear that all penguins must be fun-loving.

A more formal argument (following the implications in the graph) is as follows:

- Let p be any penguin.
- From (2), we know that p does not eat mulberries.
- From (5), we know that p must eat chestnuts.
- From (1), we know that p must be fun-loving.

Note that the above argument is true even if there are no penguins. Of course, (4) says there is at least one penguin. But we didn't use (4) for the above argument.

(f) All penguins are uncomfortable.

This statement doesn't seem to hold looking at the graph as it looks like all penguins are in fact comfortable. In fact, the statement is false in the world W we created for (a) (where p is a comfortable penguin). So the statement is not entailed by the facts.