Relations Tutorial Problems

1. Constructing a concrete relation

Construct a relation R on the set $\{1, 2, 3\}$ such that all the following are true:

- $1R2$
- R is symmetric
- R is transitive
- R is not an equivalence relation

(You are constructing just one relation which satisfies all four conditions, not *a separate relation for each condition. You can specify the relation however you want: a diagram with arrows, a table of related pairs, etc.)*

2. Discussion manual problems

Do the following problems from the discussion manual. (Note that when these problems say something like "Define a relation R on A such that ..."; they mean "*We are hereby defining* a relation R on A such that ...". In particular, it is *not* asking *you* to provide a definition.)

- 4.2 parts (a) and (b)
- 4.3 part (a), except you do not need to prove the relation is an equivalence relation.
- 4.3 part (b)

3. Abstract relation proof

Let R and S be symmetric relations on some set A. Define a relation $\sim$ on A such that $x \sim y$ if and only if xRy and $\neg(xSy)$. Prove that $\sim$ is symmetric.

4. Sorting

(This question is purposefully more open-ended than usual. Don't worry about getting the same answer as us, and move on when you don't have more to profitably discuss.) In programming, sorting a list of numbers in ascending order is sometimes called "sorting by $<$". A good sorting API will allow you to sort by user-defined relations - e.g. if you want all the odds ascending followed by all the evens ascending, this would be sorting by R where R is defined by "aRb iff either a is odd and b is even, or they have the same parity[1] and $a < b$". But there are also relations which are not usable for sorting. In general, what properties should a relation R have (or not have) in order for "sorting by R" to make sense (for example, does R need to be reflexive, symmetric, etc?)?

[1] i.e. they're both odd or both even

Solutions

1. Constructing a concrete relation

R is the relation which relates a to b for each row in this table:

a	b
1	2
2	1
1	1
2	2

(Commentary: It turns out the above solution is the only *relation that satisfies the given conditions. To construct it, we can apply the conditions one at a time as follows: First, we know that $1R2$. Then to make R symmetric, we have to also include $2R1$. Then to make R transitive, we have to include $1R1$ and $2R2$. (Remember that the definition of transitive does not require that x, y, z be distinct.) Finally we check over all the conditions again to confirm we are done. You did not have to demonstrate that this is the only possible R, but notice that if you include $3R3$ then the relation would be reflexive and thus an equivalence relation, and if you make any other additional pair of elements related, then either you will not add enough to restore symmetry and transitivity, or you will discover that everything has to be related to everything else, which is again an equivalence relation.)*

2. Discussion manual problems

4.2a)

- R is a partial order. It is reflexive because every letter has a self-loop, it is antisymmetric because no arrow has a matching arrow in the reverse direction, and we can check transitivity by confirming that for every two arrows that can be 'chained' one after another (like ARC and CRB), the corresponding third arrow (ARB) is present.

  (To check that a relation defined by an arbitrary diagram is transitive, it helps to notice that the presence *of a self-loop will never cause the relation to be non-transitive, because the following two statments are always true: $xRx \wedge xRz \to xRz$, and $xRy \wedge yRy \to xRy$. Note that the* absence *of a self-loop can sometimes make a relation non-transitive, as discussed in the lecture.)*

- T is not a partial order because it is not transitive: we have CTB and BTD, yet we do not have CTD.

4.2b) We need to show that $\preceq$ is reflexive, antisymmetric, and transitive.

- Claim: $\preceq$ is reflexive. Consider an element $(a, b) \in \mathbb{Z}^2$. We know that $a \leq a$ and $b \leq b$, so by the definition of $\preceq$, $(a, b) \preceq (a, b)$.

- Claim: $\preceq$ is antisymmetric. Consider elements $(a, b), (c, d) \in \mathbb{Z}^2$ such that $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$. Then by the definition of $\preceq$, we have $a \leq c$, $b \leq d$, $c \leq a$, and $d \leq b$. Since $a \leq c$ and $c \leq a$, we know $a = c$. By similar logic, we know $b = d$. Thus $(a, b) = (c, d)$.

  (Recall that in lecture we gave two equivalent versions of the definition for anti-symmetry; the one used here ($xRy \wedge yRx \to x = y$) is more frequently useful for proofs.)

- Claim: $\preceq$ is transitive. Consider elements $(a,b),(c,d),(e,f) \in \mathbb{Z}^2$ such that $(a,b) \preceq (c,d)$ and $(c,d) \preceq (e,f)$. Then by the definition of $\preceq$, we have $a \leq c$, $b \leq d$, $c \leq e$, and $d \leq f$. Then by transitivity of $\leq$ we get $a \leq e$ and $b \leq f$, so $(a,b) \preceq (e,f)$.

4.3a) By the definition of $\sim$, $[(1,3)]$ contains every (c,d) pair where $1+d = 3+c$, i.e. where $d-c=2$. (For example, it contains $(0,2)$ and $(1001,1003)$.) Similarly, we see that $[(0,4)]$ contains all pairs where $d-c=4$, and $[(2,4)]$ contains all pairs where $d-c=2$ (i.e. it is the same as our earlier $[(1,3)]$). More generally, for each integer k there is one equivalence class $[(0,k)] = \{(c,d) \mid d-c=k, c \in \mathbb{Z}, d \in \mathbb{Z}\} = \{(c,c+k) \mid c \in \mathbb{Z}\}$.

(Each equivalence class is a lower-left-to-upper-right diagonal line in the plane. I chose $[(0,k)]$ as my 'canonical' name for each of these diagonal lines (i.e. naming the line after the place where it crosses the y-axis), but we could just as well have taken our pick of infinitely many other possible names, including e.g. $[(-k,0)]$, naming after the place where it crosses the x-axis.)

4.3b) By the definition of $\sim$, $[2]$ contains every integer y where $4 \mid (3 \cdot 2 + 5y)$. Since $6+5y$ and $y-2$ differ by a multiple of 4 (i.e. $4(y+2)$), we see that $4 \mid (6+5y)$ iff $4 \mid (y-2)$, so $[2]$ contains all integers that are 2 more than a multiple of 4, i.e. $[2]$ is precisely $[2]_4$ (the congruence class of 2 mod 4).

Similarly, $[3]$ contains every integer y where $4 \mid (3 \cdot 3 + 5y)$. Since $9+5y$ and $y-3$ differ by a multiple of 4 (i.e. $4(y-3)$), we see that $4 \mid (9+5y)$ iff $4 \mid (y-3)$, so $[3]$ contains all integers that are 3 more than a multiple of 4, i.e. $[3]$ is just $[3]_4$.

By continuing the same logic, we see that there are only 4 equivalence classes and these are the congruence classes of 0, 1, 2, and 3 mod 4. In fact $\sim$ *is* just equivalence mod 4, i.e. $x \sim y$ if and only if $x \equiv y \pmod 4$.

We will now show $\sim$ is an equivalence relation by showing it is reflexive, symmetric, and transitive:

- Claim: $\sim$ is reflexive. Consider an integer x. $3x+5x = 8x = 4(2x)$, so $4 \mid 3x+5x$ and thus $x \sim x$.
- Claim: $\sim$ is symmetric. Consider integers x, y such that $x \neq y$ and $x \sim y$. Then by definition of $\sim$, $4 \mid 3x+5y$, i.e. $3x+5y = 4k$ for some integer k. Then $5x+3y = 8x+8y-(3x+5y) = 8x+8y-4k = 4(2x+2y-k)$. *(An alternate method: replace the previous sentence by "Then $9x+15y = 12k$, so $5x+3y = 12k-4x-12y = 4(3k-x-3y)$.")* Thus $3y+5x$ is a multiple of 4, so $y \sim x$.

  (This is probably the hardest of the properties to prove. You start with $3x+5y = 4k$ and you know you have to end with $5x+3y = 4($something$)$; it may take significant experimenting with adding equations, multiplying by constants, etc to eventually find the solution here. Don't worry about getting everything to fit together in logical order until you've first found all the pieces in your scratchwork.)
- Claim: $\sim$ is transitive. Consider integers x, y, z such that $x \sim y$ and $y \sim z$. Then by definition of $\sim$, $4 \mid 3x+5y$, and $4 \mid 3y+5z$. i.e. $3x+5y = 4k$ and $3y+5z = 4m$

for some integers k, m. Adding the equations gives us $3x + 8y + 5z = 4k + 4m$, so $3x + 5z = 4(k + m - 2y)$ and thus $x \sim z$.

3. Abstract relation proof

Let a, b be distinct elements of A such that $a \sim b$. By definition of $\sim$, aRb and $\neg(aSb)$. Since aRb and R is symmetric, we have bRa. Since $\neg(aSb)$ and S is symmetric, we have $\neg(bSa)$. *(This comes from using the definition of symmetry in its contrapositive form, i.e. for all $x, y \in A$ such that $x \neq y$, if $\neg(yRx)$, then $\neg(xRy)$.)* Finally, since we have bRa and $\neg(bSa)$, then by the definition of $\sim$, $b \sim a$.

4. Sorting

If we are building a relation $\preceq$ and we want $a \preceq b$ to mean that a should come before b in the final sorted list, then the first immediate consequence is that $\preceq$ must be antisymmetric. If instead $\preceq$ is *not* antisymmetric, that means there's a distinct a and b where $a \preceq b$ and $b \preceq a$, and in that case we need to put a before b and also b before a, which is impossible.

We also probably want transitivity. We definitely need to avoid "cycles" in the diagram, e.g. if $a \preceq b$ and $b \preceq c$ and $c \preceq a$, then again we have to put a before b and ultimately also b before a, which is impossible. Transitivity in combination with the antisymmetry from above will prevent this issue.

(To more fully answer the question, you need to decide if you want to insist that "sorting" only includes tasks where there is one right answer, such as sorting a list in ascending order, or if it also includes things like "put all the evens before all the odds, but I don't care more specifically than that". If you want the one-right-answer version, then you need an additional constraint that for every a, b, either $a \preceq b$ or $b \preceq a$. If you want the more permissive version, then you don't need this, and also you can potentially relax the transitivity requirement, as long as there are still no cycles in the diagram.)