12.1d Induction on recursive definition

The first few values are:

$x_1 = 1$
$x_2 = 7$
$x_3 = 7x_2 - 12x_1 = 7 \cdot 7 - 12 \cdot 1 = 37$
$x_4 = 7x_3 - 12x_2 = 7 \cdot 37 - 12 \cdot 7 = 175$

Proof that $x_n = 4^n - 3^n$ by induction on n:

Base: $x_1 = 1 = 4^1 - 3^1$ and $x_2 = 7 = 4^2 - 3^2$ ✓

Induction: Fix $k > 2$ and suppose (as our Inductive Hypothesis) that for any positive $i < k$, $x_i = 4^i - 3^i$. We know by the definition of the sequence that $x_k = 7x_{k-1} - 12x_{k-2}$. Since $k > 2$, $k-1$ and $k-2$ are both positive so we can apply the inductive hypothesis to get $7x_{k-1} - 12x_{k-2} = 7(4^{k-1} - 3^{k-1}) - 12(4^{k-2} - 3^{k-2})$. Finally, we simplify the right hand side as follows: $7(4^{k-1} - 3^{k-1}) - 12(4^{k-2} - 3^{k-2}) = (7 \cdot 4^{k-1} - 7 \cdot 3^{k-1}) - (3 \cdot 4^{k-1} - 4 \cdot 3^{k-1}) = 4^k - 3^k$. So $x_k = 4^k - 3^k$, which is what we needed to show.

12.2 Unrolling

a)
- $T(n) = 2T(n-1) + 3$
- $T(n) = 2(2T(n-2) + 3) + 3 = 2^2T(n-2) + 2 \cdot 3 + 3$
- $T(n) = 2^2(2T(n-3) + 3) + 2 \cdot 3 + 3 = 2^3T(n-3) + 2^2 \cdot 3 + 2 \cdot 3 + 3$.

Based on the above, we predict the general form is that for any k,

$$T(n) = 2^kT(n-k) + \sum_{i=0}^{k-1} 2^i \cdot 3$$

We want to eliminate the $T(n-k)$ using the base case definition $T(1) = 1$, so we choose $k = n-1$:

$$T(n) = 2^{n-1}T(1) + \sum_{i=0}^{n-2} 2^i \cdot 3 = 2^{n-1} + 3\sum_{i=0}^{n-2} 2^i$$

Finally, we know the closed form for $\sum_{i=0}^{n-2} 2^i$ is $2^{n-1} - 1$, so we get

$$T(n) = 2^{n-1} + 3(2^{n-1} - 1) = 4(2^{n-1}) - 3 = 2^{n+1} - 3$$

(Commentary: Notice that I did not *simplify the constant term in the unrolling step - I left it as* $3, 2 \cdot 3 + 3, 2^2 \cdot 3 + 2 \cdot 3 + 3$ *rather than* $3, 9, 21$. *This is because I frequently find it easier this way to guess the general pattern for the next step: I have no idea what to do with* $3, 9, 21$, *whereas* $3, 2 \cdot 3 + 3, 2^2 \cdot 3 + 2 \cdot 3 + 3$ *looks like a clear summation. But maybe simplifying part-way would have been even easier: you might recognize* $3 \cdot 1, 3 \cdot 3, 3 \cdot 7$ *as* $3(2^k - 1)$ *and skip the summation representation entirely. So experiment with different amounts of simplifying.)*

c)
- $T(n) = 3T(\frac{n}{3}) + 13n$
- $T(n) = 3(3T(\frac{n}{3^2}) + 13\frac{n}{3}) + 13n = 3^2T(\frac{n}{3^2}) + 13n + 13n$

- $T(n) = 3^2(3T(\frac{n}{3^3}) + 13\frac{n}{3^2}) + 13n + 13n = 3^3T(\frac{n}{3^3}) + 13n + 13n + 13n$

Based on the above, we predict the general form is that for any k,

$$T(n) = 3^k T(\frac{n}{3^k}) + k \cdot 13n$$

The base case occurs when $\frac{n}{3^k} = 1$, i.e. when $k = \log_3(n)$:

$$T(n) = 3^{\log_3(n)} T(1) + \log_3(n) \cdot 13n = 47n + 13n \log_3(n)$$

(Commentary: Remember that unrolling is not a proof *technique; this work alone is not enough to be* certain *that $T(n) = 47n + 13n\log_3(n)$, because it's possible that the pattern we noticed in the unrolling was wrong. If you want more practice with induction, you can prove that the closed forms part (b) and from this part are indeed correct.)*

Invalid recursion

- f is valid.
- g is invalid because neither case covers $n = 7$. (This also means that g is not defined for any larger value of n, e.g. $g(8)$ is undefined because its definition relies on $g(7)$.)
- h is invalid because $h(7)$ and above are undefined: $h(7)$ is defined in terms of $h(8)$, which is defined in terms of $h(9)$, etc in an *infinite* chain that never reaches a base case. (You could attempt to resolve this by reading this definition 'in reverse', i.e. if $h(n) = n + h(n+1)$ then $h(n+1) = h(n) - n$, which looks more like a valid definition. But notice that there is still no way to compute $h(7)$: you can't say $h(7) = h(6+1) = h(6) - 6$ because when $n = 6$ the "$n + h(n+1)$" case of h's definition does not apply.)
- s is invalid because both cases include $n = 7$ yet disagree on its value - does $s(7) = 2$ (from the first case), or does $s(7) = 7 + s(6) = 7 + 2 = 9$ (from the second case)?

(Note that all *of the functions are technically well-defined if you restrict the domain far enough. For example, g is a well-defined function on the domain $\{6\}$, but that is not at all a "sensible" domain for a function whose definition claims to have an $n > 7$ case.)*

12.3 Graphs and recursion

a)

k	V_k	E_k
1	2	0
2	4	4
3	6	12
4	8	24
5	10	40
6	12	60

b) $V_k = 2k$

c) $E_k = \begin{cases} 0 & \text{when } k = 1 \\ E_{k-1} + 2 \cdot V_{k-1} & \text{when } k > 1 \end{cases}$

because each graph includes all the previous graph's edges (E_{k-1}), plus a new edge from each of the (2) new vertices to each of the (V_{k-1}) old vertices. Plugging in $V_{k-1} = 2(k-1)$, we get

$E_k = \begin{cases} 0 & \text{when } k = 1 \\ E_{k-1} + 4(k-1) & \text{when } k > 1 \end{cases}$

d) We proceed by unrolling:

- $E_k = E_{k-1} + 4(k-1)$
- $E_k = (E_{k-2} + 4(k-2)) + 4(k-1) = E_{k-2} + 4((k-2) + (k-1))$
- $E_k = (E_{k-3} + 4(k-3)) + 4((k-2) + (k-1)) = E_{k-3} + 4((k-3) + (k-2) + (k-1))$

Based on the above, we predict the general form is that for any x,

$$E_k = E_{k-x} + 4\sum_{i=1}^{x}(k-i)$$

The base case occurs when $k - x = 1$, i.e. when $x = k - 1$:

$$E_k = E_1 + 4\sum_{i=1}^{k-1}(k-i) = 4\sum_{i=1}^{k-1}(k-i)$$

Finally, we can split the summation into two simpler ones and then use known closed forms for those:

$$E_k = 4(\sum_{i=1}^{k-1} k - \sum_{i=1}^{k-1} i) = 4((k-1)k - \frac{(k-1)k}{2}) = 2(k-1)k$$

14.1d Induction with Inequalities

Proof by induction on k.

Base: $f(1) = 1 + 3f(0) = 1 < 4$.

Induction: Suppose $f(j) < 4j$ for all $0 < j < k$, where $k \geq 2$. We must show that $f(k) < 4k$. First, we will show that $f(\lfloor \frac{k}{3} \rfloor) < \frac{4k}{3}$. We do this using two cases:

- Case I: $\lfloor \frac{k}{3} \rfloor = 0$. In this case, $f(\lfloor \frac{k}{3} \rfloor) = f(0) = 0 < \frac{4k}{3}$ (because $k > 0$).
- Case II: $\lfloor \frac{k}{3} \rfloor > 0$. In this case, we can apply the inductive hypothesis to get $f(\lfloor \frac{k}{3} \rfloor) < 4\lfloor \frac{k}{3} \rfloor$. Since $\lfloor k \rfloor \leq k$, this implies that $f(\lfloor \frac{k}{3} \rfloor) < \frac{4k}{3}$.
- (There is no $\lfloor \frac{k}{3} \rfloor < 0$ case since k is positive.)

Using similar case work, we can conclude that $f(\lfloor \frac{k}{5} \rfloor) < \frac{4k}{5}$ and $f(\lfloor \frac{k}{7} \rfloor) < \frac{4k}{7}$. Finally, $f(k) = k + f(\lfloor \frac{k}{3} \rfloor) + f(\lfloor \frac{k}{5} \rfloor) + f(\lfloor \frac{k}{7} \rfloor) < k + \frac{4k}{3} + \frac{4k}{5} + \frac{4k}{7} = \frac{389k}{105}$, and $\frac{389k}{105} < 4k$ because $k > 0$, QED.

Alternate solution: Alternatively, we can simplify the inductive step by proving more base cases.

Base cases: $f(1) = 1 + 3f(0) = 1 < 4$.

$f(2) = 2 + 3f(0) = 2 < 8$.

$f(3) = 3 + f(1) + 2f(0) = 4 < 12$.

$f(4) = 4 + f(1) + 2f(0) = 5 < 16$.

$f(5) = 5 + f(1) + f(1) + f(0) = 7 < 20$.

$f(6) = 6 + f(2) + f(1) + f(0) = 9 < 24$.

Inductive step: Suppose $f(j) < 4j$ for all $0 < j < k$, where $k \geq 7$. We must show that $f(k) < 4k$. Since $k \geq 7$, $\lfloor \frac{k}{3} \rfloor$, $\lfloor \frac{k}{5} \rfloor$, and $\lfloor \frac{k}{7} \rfloor$ are all at least 1, and so we can apply our inductive hypothesis to get $f(k) = k + f(\lfloor \frac{k}{3} \rfloor) + f(\lfloor \frac{k}{5} \rfloor) + f(\lfloor \frac{k}{7} \rfloor) < k + 4\lfloor \frac{k}{3} \rfloor + 4\lfloor \frac{k}{5} \rfloor + 4\lfloor \frac{k}{7} \rfloor$. Since $\lfloor x \rfloor \leq x$ for every x, we know that $f(k) < k + \frac{4k}{3} + \frac{4k}{5} + \frac{4k}{7} = \frac{389k}{105}$, which is less than $4k$ because $k > 0$.

Thus, for all $k > 0$, $f(k) < 4k$.

(Commentary: Notice that you can NOT combine the simple base case of the first solution with the simple inductive step of the second solution, because for small k (e.g. $k = 2$), $\lfloor \frac{k}{7} \rfloor = 0$ so the inductive hypothesis $f(j) < 4j$ does not apply to $f(\lfloor \frac{k}{7} \rfloor)$.)