Induction Tutorial Solutions

11.1 Simple examples

b) We proceed by induction on n.

Base: Let $n = 1$. Then $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{(1+1)}$.

Induction: Suppose (as our Inductive Hypothesis) that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for each $n \in \mathbb{Z}^+$ less than some positive integer r. Then our goal is to show $\sum_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$.

$$
\begin{aligned}
\sum_{k=1}^{r} \frac{1}{k(k+1)} &= \sum_{k=1}^{r-1} \frac{1}{k(k+1)} + \frac{1}{r(r+1)} && \text{(pulling a term out of the summation)} \\
&= \frac{(r-1)}{(r-1)+1} + \frac{1}{r(r+1)} && \text{(by the Inductive Hypothesis)} \\
&= \frac{(r-1)}{r} + \frac{1}{r(r+1)} && \text{(this and remaining steps are just algebra)} \\
&= \frac{(r-1)(r+1)+1}{r(r+1)} \\
&= \frac{r^2}{r(r+1)} \\
&= \frac{r}{r+1}
\end{aligned}
$$

Thus (by transitivity of equality) $\sum_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$, QED.

c) *This problem was moved this semester from tutorial to your PTC ProofBlocks problem, but I'm leaving the solution here in case you find it useful.*

The IH on top of the equals sign below is a shorthand for showing where we are applying the Inductive Hypothesis.

Proof by induction on n.

Base: Let $n = 0$. Then $(\sum_{i=0}^{0} i)^2 = 0 = \sum_{i=0}^{0} i^3$. ✓

Induction: Fix k and suppose that $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \cdots, k-1$. Then we get the following:

$$\begin{aligned}
(\sum_{i=0}^{k} i)^2 &= (\sum_{i=0}^{k-1} i + k)^2 \\
&= (\sum_{i=0}^{k-1} i)^2 + 2k(\sum_{i=0}^{k-1} i) + k^2 \\
&\overset{IH}{=} \sum_{i=0}^{k-1} i^3 + 2k(\sum_{i=0}^{k-1} i) + k^2 \\
&= \sum_{i=0}^{k-1} i^3 + 2k\frac{(k-1)k}{2} + k^2 && \text{(by the given hint)} \\
&= \sum_{i=0}^{k-1} i^3 + k^3 \\
&= \sum_{i=0}^{k} i^3
\end{aligned}$$

Induction complete.

11.2 Induction with congruences

Fix $a, b \in \mathbb{Z}$ and $p \in \mathbb{Z}^+$. Now we need to show $\forall n \in \mathbb{Z}^+, P(n)$, where $P(n)$ is "if $a \equiv b \pmod p$ then $a^n \equiv b^n \pmod p$". We proceed by induction on n:

Base: We need to show $P(1)$, i.e. that if $a \equiv b \pmod p$ then $a^1 \equiv b^1 \pmod p$. But this is clearly true since $a = a^1$ and $b = b^1$.

Induction: Fix z, and suppose (as our Inductive Hypothesis) that for any i with $1 \leq i < z$, $P(i)$ is true. Now we need to show $P(z)$ is true, i.e. we need to show that if $a \equiv b \pmod p$ then $a^z \equiv b^z \pmod p$.

So suppose (towards direct proof) that $a \equiv b \pmod p$. Using this fact along with $P(z-1)$ (which is true by the IH), we also know that $a^{z-1} \equiv b^{z-1} \pmod p$. Multiplying our two equivalences together gives us $a \cdot a^{z-1} \equiv b \cdot b^{z-1} \pmod p$. This in turn gives us $a^z \equiv b^z \pmod p$, QED.

(Commentary: notice that the original claim has four variables in it - a, b, n, p. It would be valid to attempt an induction proof using any of those four as the induction variable, but if you pick something other than n in this case you will discover that there is no good way to finish the proof. So as always, don't be afraid to switch tactics if your current path seems not to be working.)

11.4 A broken induction proof

(Commentary: Obviously the proof must be wrong since the claim it is proving is clearly false. While that is not enough to say where *the flaw in the proof is, it does give us a good*

place to check: $P(1)$ is true but $P(2)$ is false, so we should look at the inductive step and carefully audit its argument that $P(1) \to P(2)$.)

The argument implicitly relies on the fact that S' and S'' are not disjoint. If the sets overlap by even one horse H_*, then the proof is correct that all horses in the union are the same color, since all the horses in S' are H_*'s color and so are the horses in S''. However, consider the argument in the inductive step when $k = 2$. In this case, $S' = \{H_2\}$ and $S'' = \{H_1\}$, which are disjoint. Thus while it is true that all the horses in S' are the same color and all the horses in S'' are the same color, it is wrong for the proof to claim from this that all the horses in the union must also be the same color.

The Diagonal Robot

Let $P(n)$ be the claim "After the robot takes n steps, it is guaranteed to be at a location (x, y) with $x + y$ even." If we can prove that $P(n)$ is true for all $n \in \mathbb{N}$, then we are done, because $(0, 1)$ has an odd sum of coordinates. So we proceed by induction on n:

Base: After the robot takes zero steps, it is still at its starting location of $(1, 1)$, and $1+1$ is even.

Induction: Suppose $P(n)$ is true for $n = 0, 1, \cdots, k$. Then we need to show $P(k + 1)$, i.e. that after the robot takes $k + 1$ steps, it is still guaranteed to be at a location with even coordinate sum. So consider an arbitrary case where the robot has taken $k + 1$ steps. One step earlier, it had taken k steps, and by the inductive hypothesis, it was at a location (x, y) with $x + y$ even. So now after the $(k + 1)$st step, the only four places it can be are $(x+1, y+1)$, $(x+1, y-1)$, $(x-1, y+1)$, and $(x-1, y-1)$. In each of those cases the sum is either $x + y + 2$, $x + y$, or $x + y - 2$, so since $x + y$ is even, the new sum is also even.