Graphs and Bounding Tutorial Solutions

8.1a Paths

(Recall that a walk includes a sequence of nodes and a sequence of edges, but that usually we can give just one of those. I prefer giving the sequence of vertices, but I demonstrate the other style in the first of the bullet points below.)

- (A, F): The possible paths are 654 and 12354. Non-path walks include 11654 and 123612354. *(Recall that every path is also a walk.)*
- (F, E): The possible paths are $FDCE$ and $FDCABE$. One non-path walk is $FDCDCE$.
- (B, D): The possible paths are $BACD$ and $BECD$. One non-path walk is $BACEBECABACDFD$.
- (B, F): The possible paths are $BACDF$ and $BECDF$. One non-path walk is $BACACDF$.

8.3b Graph Connectivity

There are three connected components: the solitary node g, the solitary node h, and then everything else.

8.4 Graph Diameters

- K_n: 0 if $n = 1$, 1 otherwise. K_1 has just one vertex (which is at distance 0 from itself), so it has diameter 0. For any larger complete graph, any two distinct nodes are at distance 1 because there is an edge from every node to every other. The diameter is thus 1 (regardless of how large n is).
- C_n: $\lfloor \frac{n}{2} \rfloor$. For even n, the maximum distance is $\frac{n}{2}$, i.e. the distance between two nodes that are exactly opposite each other. For odd n, the maximum distance is still between nodes that are as close to opposite as possible, but those nodes aren't quite opposite and the two paths between them are of lengths $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. The distance between them, and hence the diameter of the graph, is the smaller of the two, $\lfloor \frac{n}{2} \rfloor$.

Thus in both even and odd cases, the diameter can be expressed as $\lfloor \frac{n}{2} \rfloor$. *(There are other ways to express this answer, including:*

$$\begin{cases} n/2 & \text{when } n \text{ is even} \\ (n-1)/2 & \text{when } n \text{ is odd} \end{cases}$$

- W_n: 1 if $n = 3$, 2 otherwise. For $n = 3$, W_n is just K_4 which as we saw above has diameter 1. For any larger n, there are non-adjacent nodes in the rim so the diameter must be larger than 1, but there is a short path from any node to any other that goes through the 'hub' of the wheel, so the diameter is 2.

8.5 Euler circuits

1. One possible circuit is $ablefijmcdkgh$.

2. No Euler circuit is possible because there is at least one node (S) with odd degree. *(Note that there* does *exist an "Euler walk" - it is possible to start from S and end at the one other odd-degree vertex. But an Euler* circuit *must start and end on the same node.)*

9.2b Proving non-isomorphism

B_1 has no node with degree 3, while B_2 does (node 3).

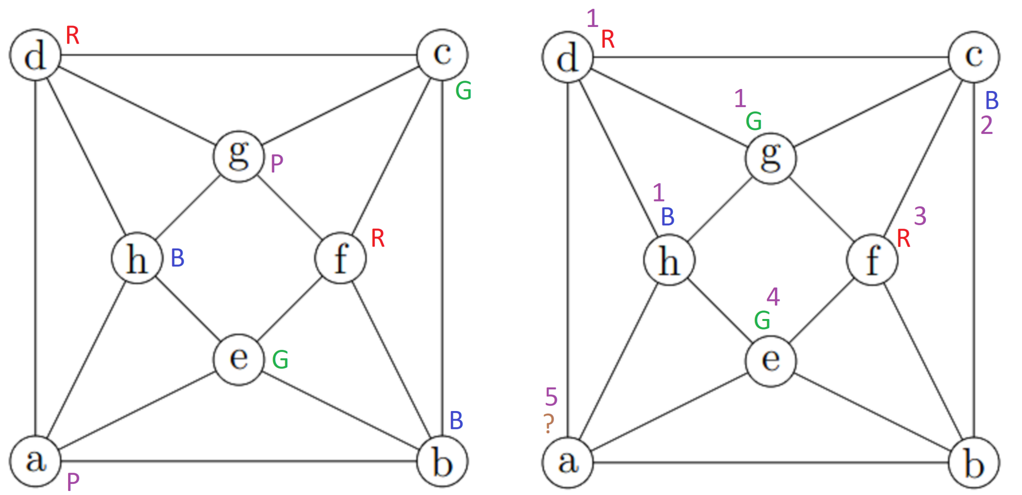
9.1b Isomorphic or not?

No isomorphism is possible: in B_1 there are two vertices of degree 3 (B and D) and they are not adjacent, while in B_2 there are also two degree-3 vertices (3 and 6) but they *are* adjacent.

(There are other features you could use to prove non-isomorphism. For example, in B_2 the three nodes of degree 4 $(1, 4, 5)$ are all adjacent to each other; B_1 also has three nodes of degree 4 (F, E, A) but there is no edge between A and F.)

9.3a Counting isomorphisms

We can permute y, p, z (3! ways to do this), v, u (2!), and a, b, c (3!); w must map to itself. There are thus $3! \cdot 2! \cdot 3! = 72$ isomorphisms.



(a) A four-coloring of the graph.

(b) There is no three-coloring. (numbers correspond to steps in the proof)

Figure 1: Problem 10.2d

10.2d Chromatic Number

Claim: the chromatic number $\chi(D)$ is 4. Proof: Figure 1a provides an upper bound of 4 by showing an explicit four-coloring, so it remains to show that the graph cannot be colored with 3 colors. We prove this as follows (see Figure 1b for a visualization): (1) Any 3-coloring must assign different colors to d, g, h; without loss of generality we call those three colors Red, Green, and Blue, respectively. (2) c is adjacent to d (Red) and g (Green), so it must be Blue. (3) f is adjacent to c (Blue) and g (Green), so it must be Red. (4) e is adjacent to f (Red) and h (Blue), so it must be Green. (5) Finally, a is adjacent to nodes of all three colors (d, h, e), so there is no possible color for a.

(Commentary: Notice that it would not *be enough to just have argued that* one particular *attempt at coloring with three colors didn't work. Instead, we argued that* every *attempt at three-coloring would run into this problem. At the beginning we do assign colors of our choice to d,g,h, but because color names are interchangeable and those three do need to be different from each other in any coloring, we haven't really made a significant choice, hence the "without loss of generality". For contrast, we could* not *have started the proof by assigning d,c,b to three different colors, because then we would not have*

addressed any colorings that may exist where d and b are given the same *color.*

This proof could also be formalized as a "proof by contradiction", but we haven't covered those yet.)

10.1b Set Equality Proofs

We will proceed by proving that each of the two sets is a subset of the other.

Subclaim: $X \subseteq Y$. Proof: Let z be an element of X. Then by definition of X, $z = 10x + 15y$ for some integers x, y. Factoring out the 5, we get $z = 5(2x + 3y)$. $2x + 3y$ is an integer since x, y are integers, so $z \in Y$. ■

Subclaim: $Y \subseteq X$. Proof: Let w be an element of Y. Then $w = 5k$ for some integer k. Then notice that $w = 10(-k) + 15k$. Since k is an integer, $-k$ is also an integer, so we see $w \in X$. ■

Since each set is a subset of the other, the two sets are equal, QED.

Additional problem: bounding vertices in 7-edge connected graph

We first prove the following lemma: A graph with n vertices has at most $n(n-1)/2$ edges. Proof: Each edge can be uniquely described by choosing one of the n possible endpoints, then one of the $n-1$ possible second endpoints, and then dividing by 2 because the order of the endpoints doesn't matter. ■

We now proceed by proving four claims:

- Lower bound subclaim 1: G has at least 5 vertices. Proof: by the lemma above, any graph with 4 or fewer vertices has at most $4 \cdot 3/2 = 6$ edges; G has 7 edges so it must have more than 4 vertices.

- Lower bound subclaim 2: It is possible for G to have 5 vertices. Proof: If we delete any 3 edges from K_5, we get a connected graph with 7 edges and 5 vertices. *(It might be better to provide one concrete example using a picture, but we'll let this abstract description slide since drawing graphs in* LaTeX *is annoying.)*

- Upper bound subclaim 1: G has at most 8 vertices. Justification *(proving this properly uses induction, which we haven't covered yet)*: If G had more than 8 vertices and only 7 edges then it wouldn't be connected. Informally, each new edge in a graph reduces the number of connected components by at most 1, so if we start with $n > 8$ vertices then after adding 7 edges we will have at least $n - 7 > 1$ connected components, so the graph will still not be connected.

- Upper bound subclaim 2: It is possible for G to have 8 vertices. Proof: G could be the line graph L_8: • – • – • – • – • – • – • – •.

Putting together all the claims above, we have that 5 is the tightest possible lower bound, and 8 is the tightest possible upper bound.

(Note that if you are not looking for the tightest bounds, there are infinitely many valid answers. For example, -100 *is a valid lower bound on the number of vertices in G: even without any of the other analysis above, we know any graph has a non-negative number of vertices, so G certainly has* at least -100 *vertices.)*