Countability Tutorial Solutions

19.1 Which Kind of Infinity?

A common fast way to show that a set *is* countable is to note that every element in the set has a finite representation. Also you may use the fact that there are no infinite sets with smaller cardinality than $\mathbb{N}$, so if you can show some set X is infinite and $|X| \leq |\mathbb{N}|$, then $|X| = |\mathbb{N}|$.

a) **Countably infinite**. In fact it's basically the definition of countably infinite - the bijection mapping it to $\mathbb{N}$ is $id_{\mathbb{N}}$.

b) **Uncountable**. The powerset of a set always has a (strictly) larger cardinality than that set. *(Or a handwavy 'solution' thinking about representations: these do not appear to all have finite representations - if I have an infinite set of naturals with no pattern, how would I possibly write down that set?)*

c) **Uncountable**. We know $\mathbb{R}$ is uncountable, and $\mathbb{R} \subseteq \mathbb{C}$.

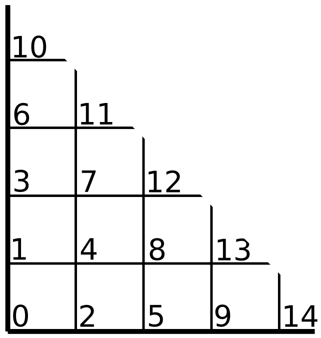
d) **Countably infinite**. There are clearly infinitely many elements (in particular at the very least there are the elements $\{0\}$, $\{1\}$, $\{2\}$, $\cdots$). To show it's *countably* infinite, we can provide a one-to-one function f mapping these to the (finite) bit strings: given S with maximum element n, return the bit string of length $n+1$ with a 1 in (0-indexed) position i iff $i \in S$. For example, $f(\{0,3,4\}) = 10011$. And we know the set of bit strings (or any other strings with a finite alphabet) is countable. *(Alternatively, thinking with representations: each $S \in X$ has a roster notation which is finite - e.g.* $\{0,3,4\}$*.)*

e) **Countably infinite**. There are clearly infinitely many elements (in particular, at the very least there are the silly books containing just "a", "aa", "aaa", $\cdots$). And it's *countably* infinitely because each book is just one (finite) string created using a fixed (finite) alphabet. *(You may be tempted to think of a book as a* list *of strings separated by spaces, but that's making it more complicated than necessary - there's no need to treat characters like space and newline any differently from a and b.)*

f) **Countably infinite**. We know $\mathbb{Q}$ is countable, and this set is a subset of $\mathbb{Q}$. *(Thinking with representations: these are reals specifically chosen to have expansions that end - i.e. representations that are finite.)*

19.2 A Curious Bijection

a)



b) Consider the values of x, y satisfying $x + y = k$.

Because we are in $\mathbb{N}$, for any such values of x and y we have that $y \geq 0$ and therefore $x \leq k$. For any value $x \leq k$, we can let $y = k - x$ to achieve $x + y = k$.

Thus, x ranges from 0 to k, and $f(x, y) = s(x + y) + x = s(k) + x$ ranges from $s(k)$ to $s(k) + k$. Remembering from lecture that $s(k) = \frac{k(k+1)}{2}$, we can also write this as:

$$\frac{k(k+1)}{2} \leq f(x, y) \leq \frac{k(k+1)}{2} + k$$

c) The preimage of 17 is $\{(2, 3)\}$. Note that $f(2, 3) = s(5) + 2 = 15 + 2 = 17$.

We can show that $(2, 3)$ is the only element in the pre-image by noting from our solution to part d) that, for all x, y, if $f(x, y) = f(2, 3)$, then $x + y = 2 + 3 = 5$. Testing all such values of x and y shows that $(2, 3)$ is the only element in the pre-image of 17.

(Alternatively, we could argue that there can't be any other element in the pre-image because, as demonstrated through parts (d) and (e), f is one-to-one.)

d) Let $k = x + y$, $l = p + q$. From the given supposition we know $k \neq l$, so without loss of generality, assume that $k < l$.

We get the following:

$$\begin{aligned}
f(x, y) &\leq \frac{k(k+1)}{2} + k && \text{[from part (b)]} \\
&= \frac{k^2 + 3k}{2} \\
&< \frac{k^2 + 3k + 2}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&\leq \frac{l(l+1)}{2} && [k < l, \text{ and } k, l \in \mathbb{Z}, \text{ so } k + 1 \leq l] \\
&\leq f(p, q) && \text{[from part (b)]}
\end{aligned}$$

This establishes $f(x, y) < f(p, q)$, so $f(x, y) \neq f(p, q)$, QED.

e) Suppose not. That is, suppose towards a proof by contradiction that $f(x, y) = f(p, q)$. Further, let $k = x + y = p + q$. Then:

$$\begin{aligned} f(x,y) &= f(p,q) \\ s(x+y)+x &= s(p+q)+p \\ s(k)+x &= s(k)+p \\ x &= p \end{aligned}$$

Since $x = p$ and $x + y = p + q$, we have that $y = q$. But we assumed that $(x, y) \neq (p, q)$, contradiction. So our initial supposition must be false, and thus instead we know $f(x, y) \neq f(p, q)$; QED.

Additional problem

Lemma: For sets A and B, there exists a one-to-one function $f : A \to B$ if and only if there exists an onto function $g : B \to A$.

Proof: See solution to the "additional tutorial problem" from the Functions week - the only difference is that now we are working with arbitrary sets instead of subsets of $\mathbb{N}$, so where that solution uses the function *minimum* (which can choose a representative from a set of naturals), we instead have to use the choice function h from the hint. $\square$

We know that by definition, there exists a one-to-one function $f : A \to B$ if and only if $|A| \leq |B|$. So now by the lemma, we've established that there exists an onto function $g : B \to A$ if and only if $|A| \leq |B|$.[1]

[1]Optional extra details: The lemma we used actually does not hold if $A = \emptyset$ and $B \neq \emptyset$ (can you find the flaw in the argument?), so our new cardinality definition would have to special-case that situation by specifying that $|\emptyset| < |B|$ for all non-empty B. Can you see why $\emptyset$ does *not* have to be treated as a special case in our normal one-to-one-based definition for cardinality?