Collections of Sets Tutorial Solutions

17.1 Power Sets 1

a) $\{\emptyset, \{rain\}, \{snow\}, \{sun\}, \{rain, snow\}, \{rain, sun\}, \{snow, sun\}, \{rain, snow, sun\}\}$

b) $\{\emptyset, \{(water, ice)\}\}$

c) $\mathbb{P}(C) - D = \{\emptyset, \{ice\}, \{water\}, \{water, ice\}\} - \{\{water\}, \{milk\}\} = \{\emptyset, \{ice\}, \{water, ice\}\}$

d) $\{\emptyset\}$

e) $2^5 + 2^3 - 1 = 32 + 8 - 1 = 39$ *(The -1 is because the empty set is in both powersets.)*

17.2a Power Sets 2

$\{\emptyset, \{Elm, Vine\}, \{Elm, Birch\}, \{Elm, Maple\}, \{Vine, Birch\}, \{Vine, Maple\},$
$\{Birch, Maple\}, \{Elm, Vine, Birch, Maple\}\}$

(There are other things in the powerset $\mathbb{P}(C)$, like $\{Elm\}$ and $\{Elm, Vine, Birch\}$, but they have odd cardinality and are thus excluded.)

17.3 Set-valued Functions

a) $\mathbb{Z} - \{10, 37\}$. *(Every integer other than 37 appears in $f(37)$, and every integer other than 10 appears in $f(10)$; the intersection contains every integer which is in both, i.e. every integer other than 10 and 37.)*

b) $\{3, 4\}$

c) $\{4, 7\}$

17.4 Partitions

a) yes

b) no, contains $\emptyset$

c) no, contains $\{\emptyset\}$ but the empty set is not a member of S

d) no, h is not in S

e) yes

f) no, a partition of S should contain sets of letters but this set contains sets of sets of letters

17.5 Counting and Combinations

a) Use "combinations with repetition" formula with $k = 11$ objects and $n = 3$ types: $\binom{11+3-1}{11} = \binom{13}{11}$. (Or $\binom{13}{2}$, or $\frac{13\cdot 12}{2\cdot 1} = 78$.) *WARNING: Make sure you understand how to re-derive the formula for combinations with repetition, using the stars and dividers picture (section 18.6 in the textbook). Blindly memorizing the final formula leaves you open to a range of off-by-one errors.*

e) $\binom{10}{1} + \binom{10}{3} + \binom{10}{5} + \binom{10}{7} + \binom{10}{9}$

17.6 A Trinomial Theorem?

a) If we were to fully expand $(x + y + z)^{27}$ and not collect like terms yet, each of the 3^{27} terms would exactly correspond to one of the possible ways to choose an x, y, or z from each of the 27 trinomials. In particular, each instance of $x^3y^{14}z^{10}$ comes from choosing a total of 3 x's, 14 y's, and 10 z's.

There are $\binom{27}{3}$ ways to choose which trinomials provide the x's. After that there are $\binom{24}{14}$ ways to choose locations for the y's, and then the choices for z's are fully determined.

Total choices: $\binom{27}{3}\binom{24}{14} = \frac{27!}{24!3!}\frac{24!}{14!10!} = \frac{27!}{3!14!10!}$

b) Using the same logic as part (a) and the fact that $27 = a + b + c$, we get $\binom{27}{a}\binom{27-a}{b} = \frac{27!}{a!(27-a)!}\frac{(27-a)!}{b!c!} = \frac{27!}{a!b!c!}$.

Additional problems: Partitions

a) $M = \{p(2), p(5), p(7), p(8), p(13), p(21)\} = \{\{2,8\},\{5\},\{7,21\},\{2,8\},\{13\},\{7,21\}\} = \{\{2,8\},\{5\},\{7,21\},\{13\}\}$. This is a partition: by inspection, it follows all 3 rules (it covers all of A, it does not contain the empty set, and all the sets are disjoint). *(Notice that duplicates are 'automatically' removed from M since M is a set. Also note that the same function p used with a different base set - e.g.* $A = \{2, 3, 6\}$ *- might not produce a partition.)*

b) $S = \{D(0), D(1), D(2), D(3), D(4), D(5), \cdots\} = \{\emptyset, \{D\}, \{A, B\}, \{C\}, \emptyset, \emptyset, \cdots\} = \{\emptyset, \{D\}, \{A, B\}, \{C\}\}$. This is not a partition because it includes $\emptyset$.

Additional problems: Set-valued functions

a) f is not one-to-one: $f(\{2\}) = f(\{2, 3\}) = \{1\}$

b) f is onto. Consider an arbitrary element T of the codomain. Let $U = \{2x \mid x \in T\}$. Then $f(U) = T$.

Additional problems: Counting

Note that this is almost *the same as our "combinations with repetition" setting - if each x_i represents the number of objects chosen to be of type i, then the only difference is that here we are looking for solutions using positive integers while in Section 18.6 of the textbook we were looking for solutions using non-negative integers.*

We do a similar stars-and-bars analysis to that of Section 18.6 of the textbook. For a positive integer solution, we will form a list of k stars and $n-1$ bars, where we have x_1 stars, then a bar, then x_2 stars, then a bar, and so on. Just as in the case of non-negative integer solutions, the integer values for the variables in a solution can be retrieved by letting x_i count the number of stars between the $(i-1)$-th bar and the i-th bar. However, in a positive integer solution, a bar cannot be at either end of the stars-and-bars diagram, nor can two bars be adjacent to each other. We can count the possible stars-and-bars diagrams corresponding to positive integer solutions as follows: we have $n-1$ bars, each of which can be placed in one of the $k-1$ positions between two stars, but no two bars can occupy the same slot. In short, we need to choose a set of $n-1$ positions out of a set of $k-1$ total possible positions, i.e., the total number of choices is $\boldsymbol{\binom{k-1}{n-1}}$.

(Alternate solution): For each i, let $y_i = x_i - 1$. Then $(x_1, \ldots, x_n)$ is a solution to $\sum_{i=1}^{n} x_i = k$ if and only if $\sum_{i=1}^{n} y_i = k - n$. Moreover, x_i is positive if and only if y_i is non-negative. So the number of positive integer solutions to $\sum_{i=1}^{n} x_i = k$ is the same as the number of non-negative integer solutions to $\sum_{i=1}^{n} y_i = k - n$, which, from Section 18.6 of the textbook, we know to be $\binom{(k-n)+n-1}{n-1} = \boldsymbol{\binom{k-1}{n-1}}$.