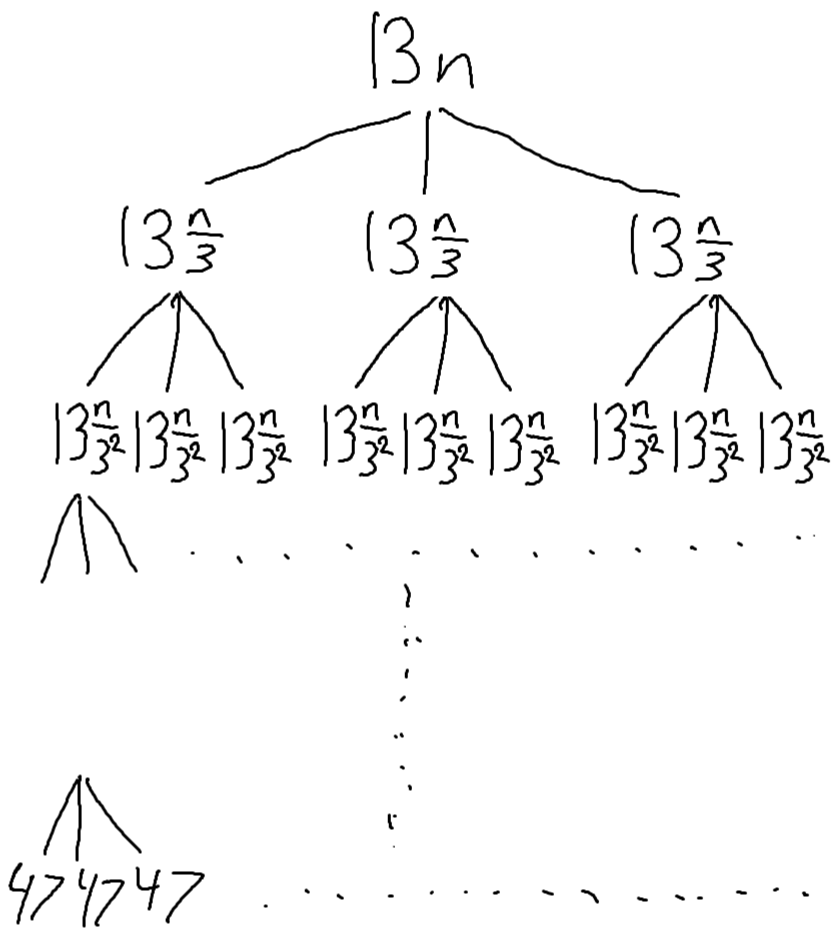
Big-O Tutorial Solutions

13.1 Recursion trees

a) Assume n is a power of 3 so that the input will always be an integer. Then we get the following tree:



The tree is described by the following table:

level	"problem size"	# nodes	work per node	total for level
0	n	1	$13n$	$13n$
1	$\frac{n}{3}$	3	$13\frac{n}{3}$	$13n$
2	$\frac{n}{3}$	3^2	$13\frac{n}{3^2}$	$13n$
3	$\frac{n}{3^2}$	3^3	$13\frac{n}{3^3}$	$13n$
...				
k	$\frac{n}{3^k}$	3^k	$13\frac{n}{3^k}$	$13n$
...				
h	$\frac{n}{3^h} = 1$	3^h	$T(1) = 47$	$47 * 3^h$

(Notice that the final row (the leaf level) follows the same pattern for problem size and number of nodes as the rows above it, but that we also know the problem size must be 1 *since that's the function's base case - this is why I've written both* $\frac{n}{3^h}$ *and* 1 *in that cell,*

and this is how we are able to solve for h. Note that the work per node and hence total for level does not *follow the pattern of the levels above it; this is why our later summation only sums through $h-1$ and then we have to add in the work in the leaves separately.)*

We have $\frac{n}{3^h} = 1$, i.e. $h = \log_3 n$, so there are $3^{\log_3 n} = n$ leaves. Thus the total work at the leaves is $n \cdot T(1) = 47n$.

From the table, the total work for all non-leaf levels is

$\sum_{k=0}^{(\log_3 n)-1} 13n = 13n \log_3 n$.

Putting it all together, our final closed form is $47n + 13n \log_3 n$.

b) We'll just describe the tree with a table instead of drawing it:

level	"problem size"	# nodes	work per node	total for level
0	n	1	3	3
1	$n-1$	2	3	6
2	$n-2$	4	3	12
3	$n-3$	8	3	24
...				
k	$n-k$	2^k	3	$3 \cdot 2^k$
...				
h	$n-h=1$	2^h	T(1) = 1	$1 \cdot 2^h$

From $n-h=1$ we get $h = n-1$, so there are $2^h = 2^{n-1}$ leaves, for a total work in the leaves of $2^{n-1} \cdot T(1) = 2^{n-1}$.

The total work for all non-leaf levels is

$\sum_{k=0}^{n-2}(3 \cdot 2^k) = 3\sum_{k=0}^{n-2} 2^k = 3(2^{n-1}-1)$.

Thus our closed form is $2^{n-1} + 3(2^{n-1}-1) = 4 \cdot 2^{n-1} - 3 = 2^{n+1} - 3$.

14.1 Induction with Inequalities

a) Proof by induction on n.

Base: At $n=8$: $8^2 = 64$, and $7 \cdot 8 + 1 = 57$ which is smaller.

Induction: Fix $k \geq 8$ and suppose (as our Inductive Hypothesis) that $n^2 > 7n+1$ for each n from 8 through k. In particular, $k^2 > 7k+1$. Then we get the following:

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&> (7k+1) + 2k + 1 && \text{[by the IH]} \\
&= 7k + (2k+1) + 1 \\
&> 7k + 7 + 1 && \text{[since } k \geq 8,\ 2k+1 \geq 17 \geq 7\text{]} \\
&= 7(k+1) + 1
\end{aligned}$$

So $(k+1)^2 > 7(k+1)+1$, QED.

(Commentary: As is frequently the case with these inequality proofs, there is a step that looks like total magic if we just read from top to bottom: how did we know to replace 2k+1 by 7?? You come up with a step like this by ***working from both ends*** *and making the expressions look as close to each other as possible - in this case, after applying the IH, we have* $(7k+1)+2k+1$ *on one end and* $7(k+1)+1$ *on the other; only after you've rearranged them to look as similar as possible (i.e. they both have a* $7k$ *term and either a* 1 *or a* 2 *term) will the 'magic' inequality step become apparent.)*

c) Proof by induction on n.

Base: We check that the claim holds for $n = 2$: $\frac{1}{2^2} = \frac{1}{4} = \frac{3}{4} - \frac{1}{2}$ ✓

Induction: Fix $k \geq 2$ and suppose that $\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq \frac{3}{4} - \frac{1}{n}$ for $n = 2, ..., k$. In particular, $\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} \leq \frac{3}{4} - \frac{1}{k}$. Then we get the following:

$$\begin{aligned}
\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{(k+1)^2} &= (\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2}) + \frac{1}{(k+1)^2} \\
&\leq (\frac{3}{4} - \frac{1}{k}) + \frac{1}{(k+1)^2} && \text{[by IH]} \\
&= \frac{3}{4} + \frac{1}{(k+1)^2} - \frac{1}{k} \\
&= \frac{3}{4} + \frac{k}{k(k+1)^2} - \frac{(k+1)^2}{k(k+1)^2} \\
&= \frac{3}{4} - \frac{k^2+k+1}{k(k+1)^2} \\
&< \frac{3}{4} - \frac{k^2+k}{k(k+1)^2} && \text{[since } k > 0,\ k(k+1)^2 > 0\text{]} \\
&= \frac{3}{4} - \frac{1}{k+1}
\end{aligned}$$

Thus $\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{(k+1)^2} \leq \frac{3}{4} - \frac{1}{k+1}$, QED.

Exponential vs Factorial

1. We want to show there are positive reals k, c such that $\forall n \geq k, 0 \leq 2^n \leq c \cdot n!$. Let $k = 4$ and $c = 1$. Then it remains to show that $\forall n \geq 4, 0 \leq 2^n \leq n!$. This follows from Claim 50 in the textbook.

2. This statement is false. As a counterexample, consider $f(n) = 2^n$ and $g(n) = 1$. Then $f(n)$ is $O(2^n)$ and $g(n)$ is $O(n!)$, but $f(n)$ is not $O(g(n))$. *(Commentary: Informally, "$g(n)$ is $O(n!)$" provides an upper bound on how fast g can grow, but it does not provide a lower bound.)*

Transitivity of big-O

Fix f, g, h, and assume towards a direct proof that $f(n)$ is $O(g(n))$ and $g(n)$ is $O(h(n))$. Then by definition of big-O, there are (positive real) k_0, c_0 such that $\forall n \geq k_0, 0 \leq f(n) \leq c_0 g(n)$, and also k_1, c_1 such that $\forall n \geq k_1, 0 \leq g(n) \leq c_1 h(n)$. Now we want to show there are k, c such that $\forall n \geq k, 0 \leq f(n) \leq ch(n)$.

Let $k = max(k_0, k_1)$ and $c = c_0 c_1$. Then we need to show $\forall n \geq max(k_0, k_1), 0 \leq f(n) \leq (c_0 c_1) \cdot h(n)$. To do this, fix $n \geq max(k_0, k_1)$. Then we have $0 \leq f(n)$ (since $n \geq max(k_0, k_1) \geq k_0$), and also:

$$\begin{aligned}
f(n) &\leq c_0 g(n) && \text{(since } n \geq max(k_0, k_1) \geq k_0\text{)} \\
&\leq c_0(c_1 h(n)) && \text{(since } n \geq max(k_0, k_1) \geq k_1\text{)} \\
&= (c_0 c_1) \cdot h(n) && \text{(rearrange)}
\end{aligned}$$