Sets and Modular Arithmetic Tutorial Problems

1. Congruence classes of perfect squares
   a) Compute \{[x^2]_4 \mid x \in \mathbb{Z}\}. (That is, rewrite the set into a simpler form that lists all the elements explicitly.)

   b) Notice that, for any \(k\), \([a]_k \neq [b]_k\) implies \(a \neq b\). (Do you see why this is true?) Using this fact and the result from part (a), prove that for all integers \(x\) and \(y\), \(x^2 + y^2 \neq 4000003\). (Do not use a calculator.)

2. Sets warmup
   Consider the following sets: \(A = \{2\}, B = \{A, \{4, 5\}\}, C = B \cup \emptyset, D = B \cup \{\emptyset\}\).
   a) Which of the sets have more than two elements?
   b) Which of the following are true:
      - \(2 \in A\), \(2 \in B\), \(\{2\} \in A\), \(\{2\} \in B\), \(\emptyset \in C\), \(\emptyset \in D\)
      - \(\emptyset \subseteq A\), \(\{2\} \subseteq A\), \(\{2\} \subseteq B\)

3. Cartesian product
   a) Find an example of sets \(A\) and \(B\) such that \(A \times B = B \times A\). Then find a second such pair of sets; try to make this second example feel different from your first, e.g. don’t just rename some elements.
   b) Consider the following incomplete statement:
      \[
      \text{For sets } A \text{ and } B, \text{ if } \underline{\text{______________}} \text{ then } A \times B \neq B \times A.
      \]
      Create a true claim by filling in the blank with a statement about \(A\) and \(B\) that does not mention Cartesian products. Try to make the strongest possible claim, i.e. ideally your statement should still be true even if we replaced the “if-then” by an “if and only if”. If you have extra time, also prove your claim. Hint: two sets are not-equal if and only if there exists an element that is in one but not the other.