CS 173 Discussion Problems

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1.   Proofs and Logic

1.1   Negations and contrapositives

Negate the following statements, moving all negations (e.g. "not") onto individual predicates/propositions. You may use shorthand to do intermediate work, but your final answer should be in English. Similarly, construct the contrapositive of each statement.

(a) If my plant is dead, then I didn't water it or I left it in the dark.

(b) If vampires exist, then there is a city c such that c is full of vampires and c does not have a blood bank.

(c) For every martian w, if w is green, then w is tall or w is ticklish.

(d) For any house h, for any dog d, d does not live at h or h has a supply of dog food.

(e) For every movie m, if m is a fantasy movie and m is popular, then m has a cute lead actor and m has a big special effects budget.

1.2   Direct proof and disproof

For each claim, prove it using direct proof or disprove it using a concrete counterexample.

(a) For any integer k, if k is odd, then k^3 is odd.

(b) For any integers p and q, $(p + q)^2 = p^2 + q^2$.

(c) For any real numbers w, x, y, and z, if $w < x$ and $y < z$ then $wy < xz$.

(d) For all real numbers x and y, where $x \neq 0$, if x and $\frac{y+1}{3}$ are rational, then $\frac{1}{x} + y$ is rational.

1.3 Variations on direct proof

Prove the following claim. Your proof should divide into cases based on the sign of $|x + 7|$.

(a) For any integer x, if $|x + 7| > 8$, then $|x| > 1$.

    Prove the following claims by contrapositive. Begin your proof by explicitly writing out the contrapositive. Then use direct proof to prove the contrapositive.

(b) For all real numbers x and y, if $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.

(c) For all integers m and n, if mn is even, then m is even or n is even.

(d) For all real numbers x, if $x^2 - 3x + 2 > 0$, then $x \geq 2$ or $x < 1$.

(e) For any integers m and n, if $7m + 5n = 147$, then m is odd or n is odd.

1.4 Direct proof with inequalities

Prove the following claims. Be careful as you manipulate inequalities, e.g. check signs.

(a) For any integer k, if $k > 4$ then $2k + 1 < k^2$.

(b) For any integer k, if $k > 4$ and $k^2 < 2^k$, then $(k + 1)^2 < 2^{k+1}$. (Hint: use the previous result.)

(c) For any integers m and k, if $0 < \frac{1}{k} < m$ then $\frac{m}{m^2+1} < k$.

1.5   Logic operators

The late 19th century philosopher Charles Peirce (rhymes with 'hearse,' not 'fierce') wrote about a set of logically dual operators and, in his writings, coined the term 'Ampheck' to describe them. The two most common Ampheck operators, the Peirce arrow (written $\downarrow$ or $\perp$ or $\overline{\vee}$ by different people) and the Sheffer stroke (written $\uparrow$ or $|$ or $\overline{\wedge}$ by different people), are defined by the following truth table:

p	q	$p \uparrow q$	$p \downarrow q$
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

1. (4 points) The set of operators $\{\wedge, \vee, \neg\}$ is *functionally complete*, which means that *every* logical statement can be expressed using only these three operators. Is the smaller set of operators $\{\vee, \neg\}$ also functionally complete? Explain why or why not.

2. (4 points) Express $\neg p$ using only the Sheffer stroke operation $\uparrow$.

3. (5 points) Express $p \vee q$ using only the Sheffer stroke operation $\uparrow$. Justify your answer (e.g. using a truth table).

4. (3 points) Explain why the set of operators $\{\uparrow\}$ is functionally complete.

5. (4 points) Express the Sheffer stroke operation $p \uparrow q$ using only the Peirce arrow $\downarrow$ operation. Explain why the set of operators $\{\downarrow\}$ is functionally complete.

2.   Number Theory

2.1   Modular arithmetic

When doing computations in modular arithmetic, organize your work so that intermediate results are kept small. If you're working in base k, your final result should be in the form $[n]$ where $0 \leq n < k$.

(a) In $\mathbb{Z}_{15}$, what are some values in the congruence class of $[14]$.

(b) In $\mathbb{Z}_{15}$, find the value of $[7] + [14] * [3]$.

(c) Find the first six powers of $[5]$ in $\mathbb{Z}_7$. That is compute $[5]^1$, $[5]^2$, and so on up to $[5]^6$.

(d) Calculate the value of $[9]^{12}$ in $\mathbb{Z}_{11}$. (Hint: try repeated squaring.)

2.2   Thinking about number theory

(a) Is there an integer x satisfying both congruences simultaneously? $x \equiv 7 \pmod 9$
$x \equiv 5 \pmod{12}$

(b) Is there an integer x satisfying both congruences simultaneously? $x \equiv 5 \pmod 6$
$x \equiv 3 \pmod{10}$

(c) Find an integer solution for the equation $5m + 13n = 1$. Is it the only solution or are there others?

2.3 Thinking about gcd

Are the following claims true or false? Give a counter-example (if false) or an informal explanation (if true).

(a) For any positive integers p, q, and r, if $\gcd(p, q) = 1$ and $\gcd(q, r) = 1$, then $\gcd(p, r) = 1$.

(b) For any positive integers p, q, and r, if $\gcd(p, q) = 1$ and $\gcd(p, qr) = 1$, then $\gcd(p, r) = 1$.

(c) For any positive integers d, n, m, and p, if $n \equiv m \pmod{k}$ then $d^n \equiv d^m \pmod{k}$

(d) For any positive integers a, b, c, if $a \mid bc$ and $\gcd(a, b) > 1$, then $a \mid c$.

2.4 Proof using the divides relation

Prove the following claims directly from the definition of "divides":

(a) The divides relation is transitive, i.e. for any integers a, b, and c, if $a \mid b$ and $b \mid c$, then $a \mid c$.

(b) For any integers p, q, and r, p non-zero, if $p \mid 3q$ and $3q \mid r$, then $p \mid 3q + r$.

2.5 Proofs with congruence mod k

(a) Prove that, for all integers x, y, p, q and m, with $m > 0$, if $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$, then $(x^2 + y^2) \equiv (p^2 + q^2) \pmod{m}$.

(b) Prove that, for all integers x, y, p, q and m, with $m > 0$, if $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$, then $x^2 + xy \equiv p^2 + pq \pmod{m}$.

(c) Show that if x, y and m are integers with $m \geq 2$, then if $x \equiv y \pmod{m}$ then $gcd(x, m) = gcd(y, m)$.

2.6　Euclidean Algorithm

Recall that the Euclidean Algorithm is a quickly converging method of determining the greatest common denominator (GCD) of two numbers. The algorithm has the following pseudocode:

```
gcd(a,b: positive integers)
    x := a
    y := b
    while (y > 0) do:
        r := remainder(x,y)
        x := y
        y := r
    end while
    return x
```

Note that this pseudocode handles only positive integers. To adapt it for negative integers, take the absolute value of the inputs first.

Trace the Euclidean algorithm on the the following pairs of integers by drawing a table of values for x, y, and r

(a) 1224 and 850

(b) 2639 and 4176

3. Sets

3.1 Set Builder Notation

(a) Compute $\{(x, y) \in \mathbb{Z}^2 : x \geq 0 \text{ and } y \geq 0 \text{ and } x + y = 3\}$.

(b) Compute $\{x \in \mathbb{Z} : -20 \leq x \leq 20 \text{ and } x \equiv 2 \pmod{7}\}$.

(c) Compute $\{|x| : x \in \mathbb{Z} \text{ and } -2 \leq x \leq 7\}$.

(d) Compute $\{(2x, x^3) : x \in \mathbb{R} \text{ and } x^2 = 2\}$.

3.2 Concrete Subset Proof

Let $A = \{(p, q) \in \mathbb{R}^2 : p^2 + q^2 \leq 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$. Prove that $A \subseteq B$, by choosing a representative element from the smaller set and showing that it is in the larger set.

3.3 Abstract Subset Proofs

Prove the following set containments and show, using a **concrete counterexample**, that the reverse containment does not hold.

(a) $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

(b) $(A - C) - (B - C) \subseteq (A - B)$.

4. Relations

Reflexive: For all $x \in X$, xRx.

Irreflexive: For all $x \in X$, $x \not R x$.

Not Reflexive: There exists $x \in X$ such that $x \not R x$

Symmetric: For all $x, y \in X$ such that $x \neq y$, if xRy, then yRx

Antisymmetric:

> For all $x, y \in X$ with $x \neq y$, if xRy, then $y \not R x$ (useful for intuition!)
> For all $x, y \in X$, if xRy and yRx, then $x = y$ (useful for proofs!)

Not Symmetric: There exist $x, y \in X$ with $x \neq y$ such that xRy and $y \not R x$

Transitive: For all $a, b, c \in X$, if aRb and bRc, then aRc

Equivalence Relations are reflexive, symmetric, and transitive.

Partial Orders are reflexive, antisymmetric, and transitive.

Strict Partial Orders are irreflexive, antisymmetric, and transitive.

4.1 Relation properties

(a) Define a relation $\sim$ on intervals of the real line by $(x, y) \sim (p, q)$ if and only if $y = p$ or $x = q$. Is $\sim$ an equivalence relation? Briefly justify your answer.

(b) Define a relation $\sim$ on the positive real numbers such that $x \sim y$ if and only if $xy = 1$. Is $\sim$ reflexive, irreflexive, both, or neither? Is $\sim$ transitive? Briefly justify your answers.

4.2   Partial Orders

(a) The directed graphs below define two relations, R and T, on the set $\{A, B, C, D\}$. Is R a partial order? Is T a partial order? Justify your answers.
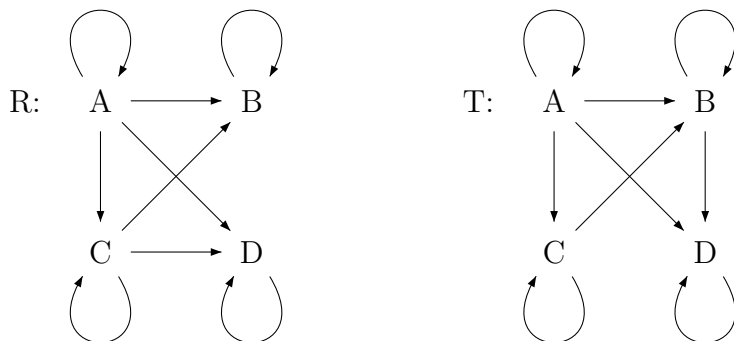
(b) Define a relation $\preceq$ on the 2D plane ($\mathbb{Z}^2$) such that $(x, y) \preceq (p, q)$ iff $x \leq p$ and $y \leq q$. Prove that $\preceq$ is a partial order.

(c) Define a relation $\preceq$ on pairs of closed intervals of the real line such that $[a, b] \preceq [c, d]$ iff $b \leq c$. Prove that $\preceq$ is antisymmetric. Assume that the first endpoint of a closed interval must be less than or equal to the second endpoint.

4.3   Equivalence Relations

For each relation, find what's in the specified equivalence classes by substituting concrete values into the definition of the relation. Then write a general description of all the equivalence classes for the relation. Finally prove that the relation is an equivalence relation.

(a) Define $\sim$ on the 2D plane $\mathbb{Z}^2$ by $(a, b) \sim (c, d)$ iff $a + d = b + c$. What's in $[(1, 3)]$? $[(0, 4)]$? $[(2, 4)]$?

(b) Define $\sim$ on $\mathbb{Z}$ such that $x \sim y$ iff $4 \mid 3x + 5y$. What's in $[2]$? $[3]$?

(c) Define $\sim$ on $\mathbb{Z}^+$ by $n \sim m$ iff $\{p \in P : p \mid n\} = \{p \in P : p \mid m\}$, where P is the set of primes. What's in $[12]$? $[18]$? $[20]$?

5.   Functions and Onto

5.1   Nested Quantifiers

Explain why each of the following propositions is true or false:

(a) $\forall x \in \mathbb{R}, \; \exists y \in \mathbb{R}, y^3 \leq x$

(b) $\exists x \in \mathbb{N}, \; \forall y \in \mathbb{N}, xy = x$

(c) $\forall x \in \mathbb{R}, \; \exists y \in \mathbb{Q}, |x - y| \leq 0.01$

(d) $\exists y \in \mathbb{Q}, \; \forall x \in \mathbb{R}, |x - y| \leq 0.01$

(e) $\exists x \in \mathbb{N}, \; \forall y \in \mathbb{N}, \mathrm{GCD}(x, y) = 1$

(f) $\forall x \in \mathbb{Z}, \; \exists y \in \mathbb{Z}, x = y^2$

(g) $\forall x \in \mathbb{Q}, \; \exists (w, y) \in \mathbb{Z}^2, x = \frac{w}{y}$

(h) $\exists (w, y) \in \mathbb{Z}^2, \; \forall x \in \mathbb{Q}, x = \frac{w}{y}$

5.2   Meditating on the Definition of Onto

Consider a function $f : A \to B$. Which of the following nested quantifier statements states that f is onto? What mathematical concepts (if any) do the other three statements represent?

(a) $\forall b \in B,\ \exists a \in A,\ f(a) = b$

(b) $\forall a \in A,\ \exists b \in B,\ f(a) = b$

(c) $\exists b \in B,\ \forall a \in A,\ f(a) = b$

(d) $\exists a \in A,\ \forall b \in B,\ f(a) = b$

5.3   Concrete Onto Proofs

Prove that the following functions are onto:

(a) $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 17 - 2x$

(b) $f : \mathbb{Z}^2 \to \mathbb{Z}$ by $f(x, y) = xy + 27$

5.4   Abstract onto proof

Suppose that $f : \mathbb{N} \to \mathbb{N}$ is an onto function. Let's define $g : \mathbb{N}^2 \to \mathbb{N}$ such that

$$g(x, y) = f(x)f(y)$$

Prove that g is onto.

6.   Function Properties

Determine whether each function is onto and/or one-to-one. Briefly explain why it is or give a concrete counter-example showing why it is not. Warning: check each definition to make sure the function is properly defined, e.g. exactly one output for each input, output values all lie in declared co-domain.

(a) $b : \mathbb{Z} \to \mathbb{Z}$ by $b(n) = 2^n$

(b) $f : \mathbb{C} \to \mathbb{R}$ by $f(x + yi) = x + y$

(c) $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3 + 7$

(d) $h : \mathbb{N}^2 \to \mathbb{N}$ by $h(x, y) = 2^x 3^y$

(e) $k : [0, 1] \to \mathbb{R}^2$ by $k(x) = x(3, 4) + (1 - x)(1, 2)$, where the product of a number a and a pair (x, y) is defined to be (ax, ay).

(f) $l : \mathbb{R} \to \mathbb{R}$ by $l(x) = \lfloor x \rfloor$

(g) $m : \mathbb{N}^2 \to \mathbb{N}$ by $m(x, y) = x - y$

(h) $p : \mathbb{Z}^2 \to \mathbb{Z}$ by $p(x, y) = xy$

(i) $q : \mathbb{Q} \to \mathbb{R}$ by $q(x) = x$

Can you make some of these functions one-to-one or onto by changing their domain and/or co-domain?

7. Functions and One-to-one

When proving that a function f is one-to-one, use the outline where you assume that $f(x) = f(y)$ and show that $x = y$. Do not use facts about derivatives or increasing functions.

7.1 Concrete One-to-one Proofs

Prove that each of the following functions is one-to-one.

(a) $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2^{x+1}$

(b) $h : \mathbb{N} \to \mathbb{Z}$ by $h(x) = x^2 + 27$

7.2 Abstract One-to-one Proof

Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is a one-to-one function. Let's define $g : \mathbb{Z} \to \mathbb{Z}^2$ such that

$$g(x) = (\ 2f(x),\ |f(x)|\ )$$

Prove that g is one-to-one.

7.3 Abstract proof using Composition

(a) Suppose that A,B, and C are sets and $f : B \to C$ and $g : A \to B$ are functions. Prove that if $f \circ g$ is onto and f is one-to-one, then g is onto.

(b) Give a concrete counter-example (involving small sets!) showing why the assumption that f is one-to-one is necessary in (a).

7.4 Permutations

(a) How many distinct strings can be formed by the letters in the word BOOTHBAY?

(b) Central Plains Pizza offers three choices for pizza diameter, two choices for crust thickness, and 12 choices of toppings. Suppose that you limit yourself to 5 or fewer toppings. How many different pizzas can you construct?

(c) In the new SCOFF programming language, variable names are strings of length between 1 and 10 (inclusive) lower-case ASCII letters (of which there are 26) where adjacent letters cannot be the same. So, 'badtb' and 'pwcycsi' are both fine, but 'abba' is bad (two adjacent letters are the same) as are 'babybabybaby' (too long), 'rolo37' (contains numbers), and 'VeryFunny' (not all lowercase). How many distinct variable names are there? Express your answer using summation notation.

7.5 Pigeonhole Principle Proofs

(a) Let d be a positive integer. Prove that in any set $A = \{x_1, ..., x_{d+1}\}$ of $d+1$ integers there will be a pair of integers x_i and x_j which are congruent modulo d.

(b) What is the maximum integer k such that, no matter how 18 people are seated in a row of 25 seats, there are k consecutive occupied seats? Prove it using the Pigeonhole Principle.

(c) A software engineer slept for 61 hours over 10 nights. Assuming that the engineer sleeps precisely an integer number of hours each night show that on some pair of consecutive nights she slept at least 13 hours.
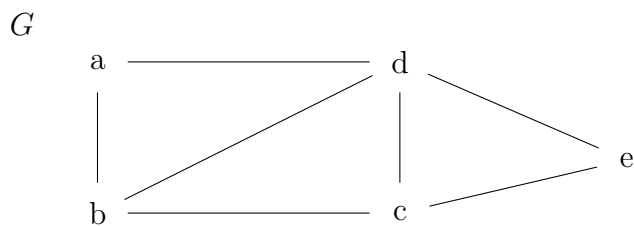
8. Graph terminology

8.1 Paths

(a) For each pair of nodes, describe a path from the first vertex to the second. You may use the given edge labels to name the edges. Give several different walks between the same two nodes.
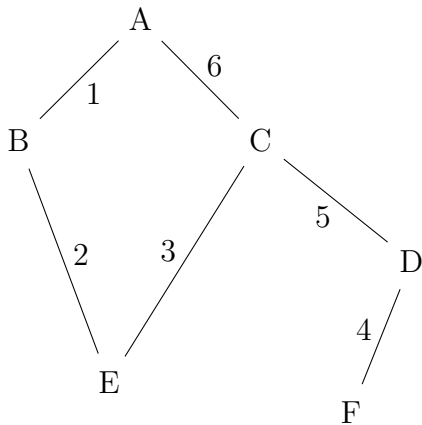
Vertex pairs to consider: (A,F), (F,E), (B,D), (B,F)

(b) List all the paths from b to e in graph G below.

8.2 Cycles

How many different cycles are in the graph K_4?

(a) Count the number of cycle subgraphs, i.e. don't worry about where each cycle starts/ends.

(b) Count the number of different walks (i.e. ordered lists of nodes with a specific starting point) that are cycles.

8.3 Graph Connectivity

(a) Is each of these graphs connected? If not, list the nodes in each connected component.

G_1:

G_2:
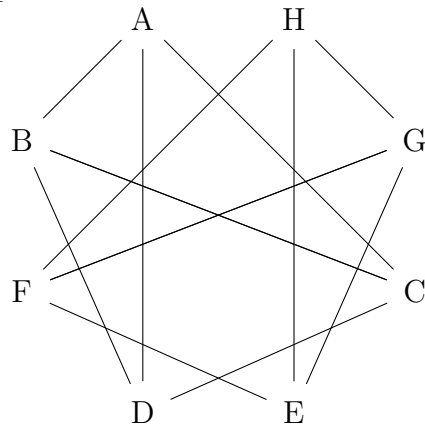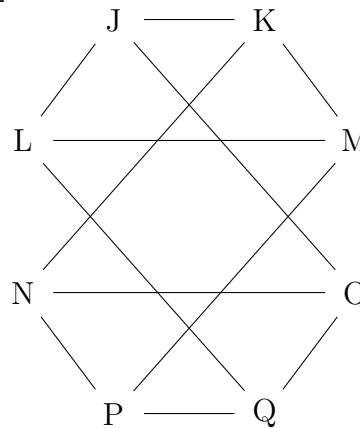
(b) How many connected components does this graph have?
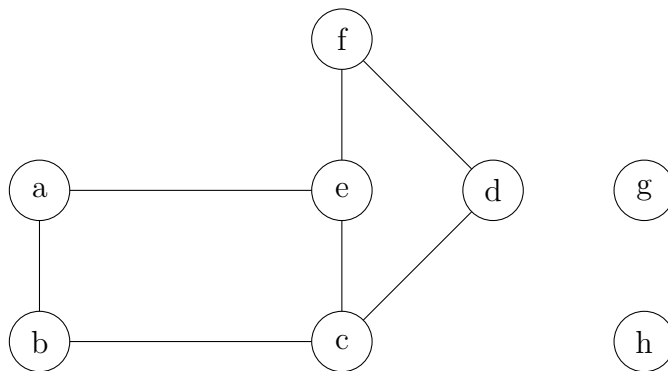
8.4 Graph Diameters

Recall that for a connected, simple graph G we define the **distance** between any two nodes v_i and v_j as the number of edges on the *shortest* path between them. Then the **diameter** G is the *maximum* distance between *any* pair of nodes in G.

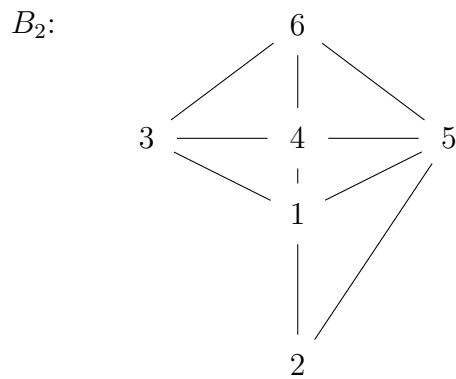Find the diameters of K_n, C_n, and W_n.

8.5 Euler circuits

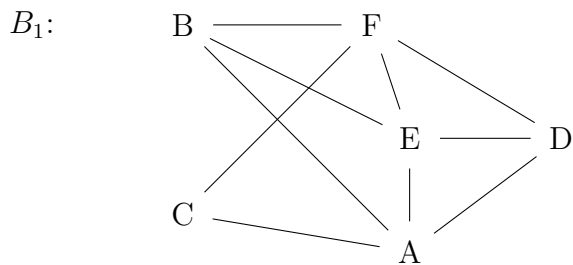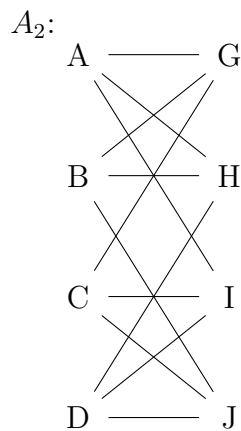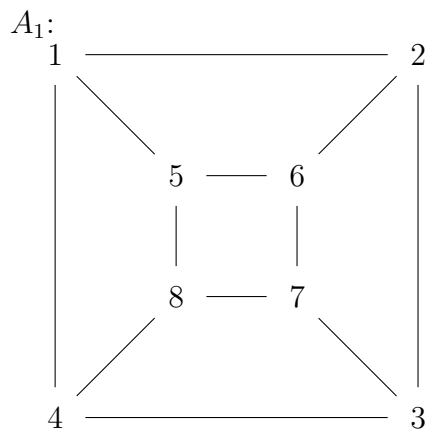Find an Euler circuit in each graph beginning at S, or explain why this isn't possible.
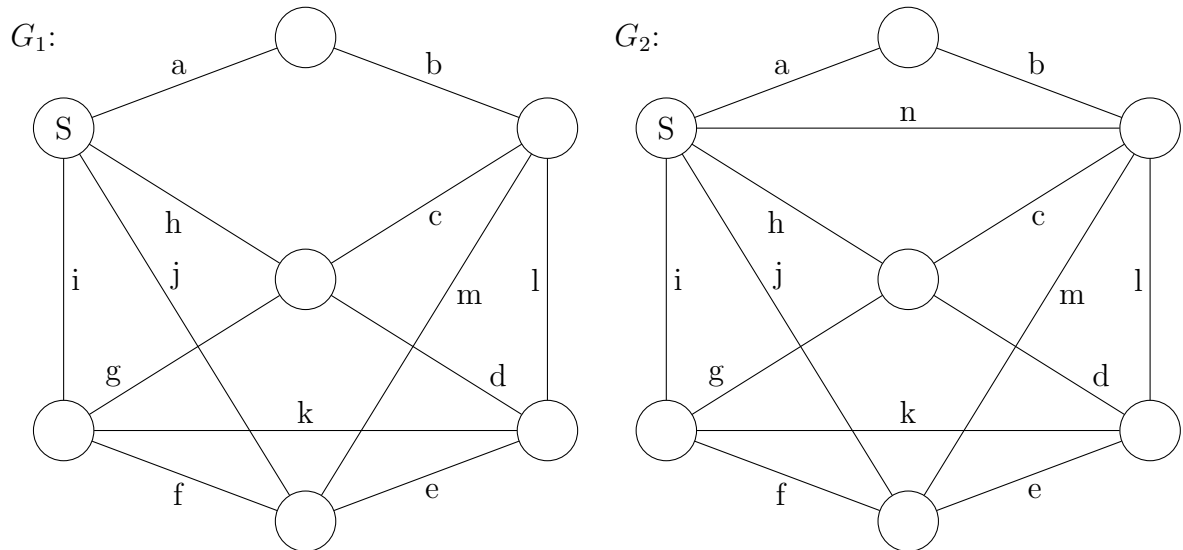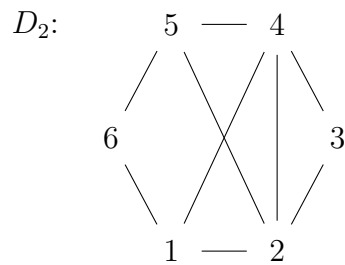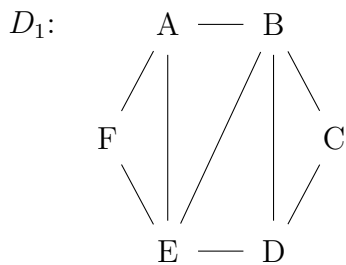
9. Graph isomorphsim

9.1 Isomorphic or not?

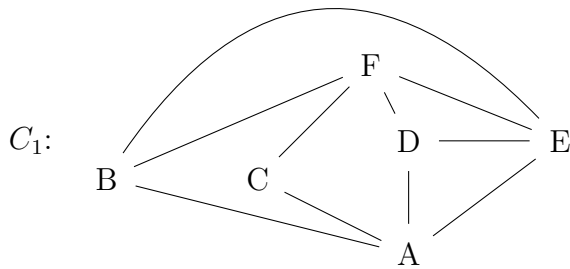Give an isomorphism between the two graphs or briefly explain why this is not possible.

A_1:

A_2:

B_1:

B_2:

9.2 Proving non-isomorphism

Prove that the two graphs in each pair are *not* isomorphic:

A_1:

A_2:

B_1:

B_2:

C_1:

C_2:

D_1:

D_2:
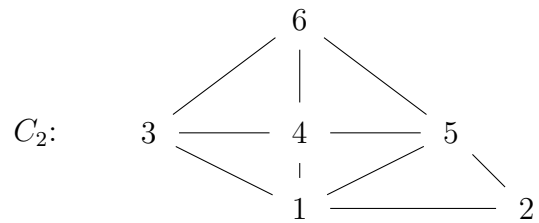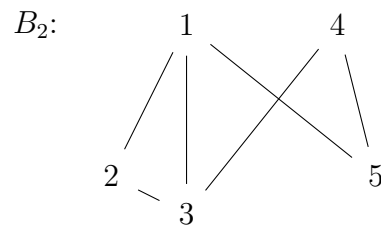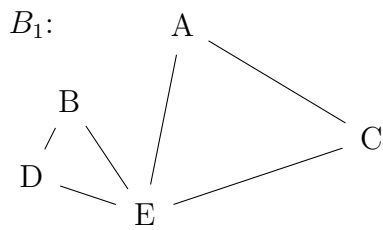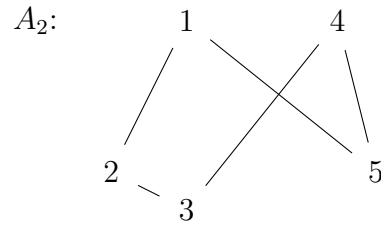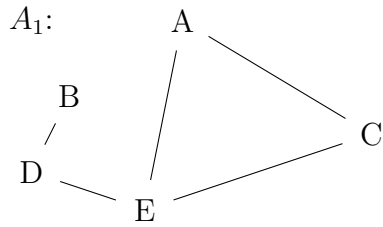
9.3 Counting isomorphisms

(a) How many isomorphisms are there from H to itself? (Notice that H contains all nine nodes.)

H:
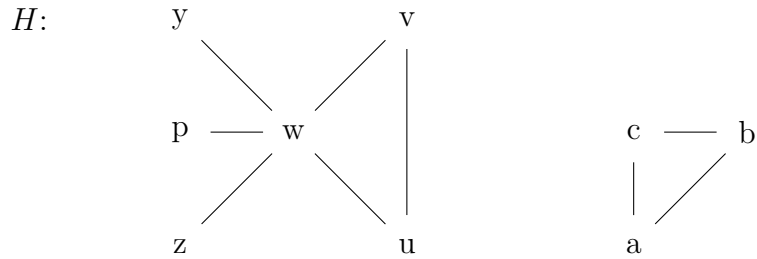
(b) How many isomorphisms are there from G to itself? (Notice that G contains all ten nodes.)

G:
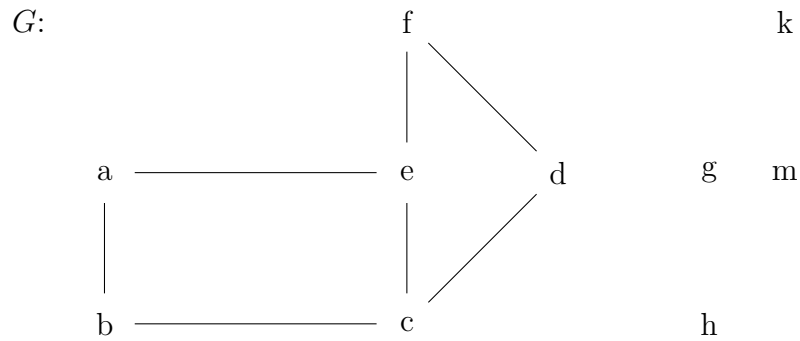
10.   Two-way Bounding

10.1   Set Equality Proofs

Prove that the following pairs of sets are equal. Or, if you are short on time, outline the proof. That is, write the main structure of the proof, and also apply the definitions of the two sets A and B, but leave out the algebra detail required to connect one definition to the other.

(a) $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 256\}$ and $B = \{(16\cos t, 16\sin t) \mid t \in \mathbb{R}\}$

(b) $X = \{10x + 15y : x, y \in \mathbb{Z}\}$ and $Y = \{n \in \mathbb{Z} : n = 5k \text{ for some } k \in \mathbb{Z}\}$

10.2   Chromatic Number

Recall that the justification that a particular chromatic number is valid requires bounding the number from above *and* below. Therefore you must give an *explicit* coloring to produce an upper bound *and* produce a valid argument that no smaller number of colors will work to produce a lower bound.

The argument justifying the lower bound often involves finding a copy of K_n (where n is the chromatic number you are attempting to validate) as a subgraph. Sometimes, however, you have to work through the space of possible $n-1$ colorings by hand and show that none of them work.

Find and justify the chromatic numbers for each of the following graphs.
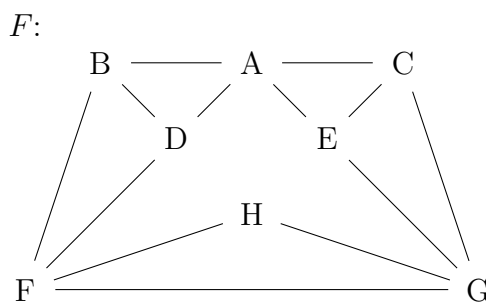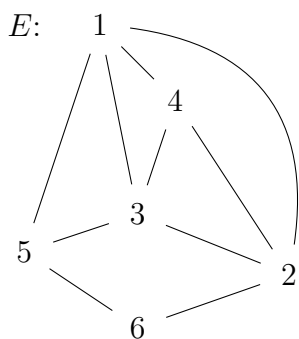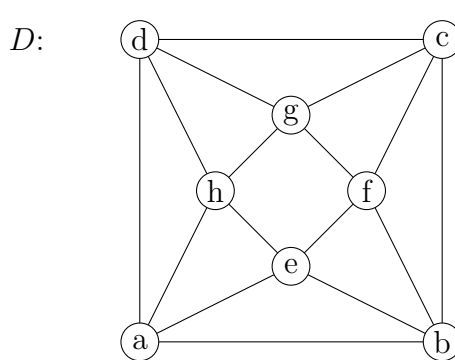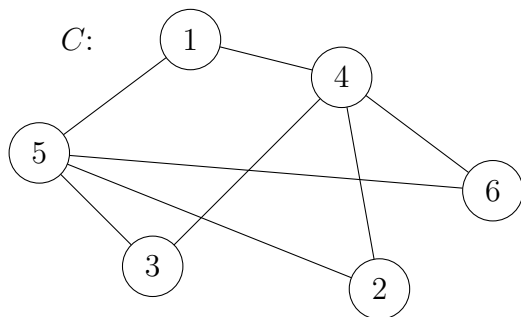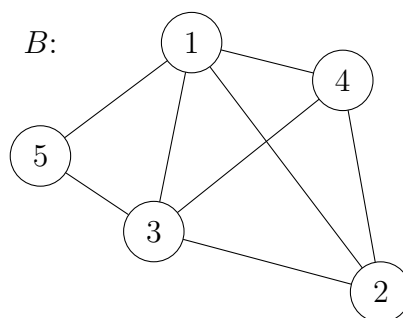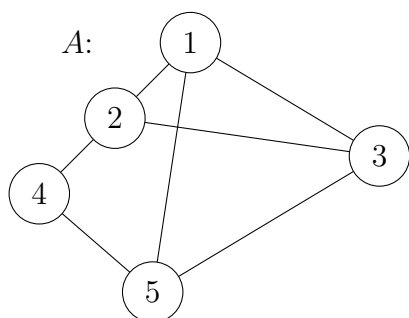
11. Induction

11.1 Simple examples

Prove the following formulas using induction:

(a) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (for all positive integers)

(b) $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ (for all positive integers)

(c) $\left(\sum_{i=0}^{n} i\right)^2 = \sum_{i=0}^{n} i^3$ (for all natural numbers)

(d) $(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$ (for all natural numbers)

Hint for (c): you can use the fact that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Hints for (d): i is the square root of -1; look up the formulas for the sin and cosine of the sum of two angles.

11.2 Induction with congruences

We've proved that if $a \equiv b \pmod{p}$ and $c \equiv d \pmod{p}$, then $a + c \equiv b + d \pmod{p}$ and $ac \equiv bd \pmod{p}$, for any integers a, b, c, and d and any positive integer p. Using one or both of these facts and induction, prove the following claim:

For any integers a and b and any positive integers n and p, if $a \equiv b \pmod{p}$, then $a^n \equiv b^n \pmod{p}$.

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11.3 Geometrical strong induction

Isaiah is making cupcakes and wants to put a single square of chocolate on top of each. So he is breaking up larger chocolate bars (e.g. 4 squares by 6) into their individual squares. Each break divides the bar in two along a straight line.

For example, we might divide a 4×3 bar into a 4×2 bar and a 4×1 bar. We might then break the larger piece in the other direction, so as to get two 2×2 bars and the 4×1 bar. Nine more breaks are required to finish the process.

Isaiah discovers that an $n \times m$ bar always seems to require $nm - 1$ breaks, apparently regardless of which sequence of directions you choose for the breaks. Use induction to prove that he's right.

Hint: Your induction variable should be a single integer that measures the size of the bar. Be sure to explain what this variable is (in terms of the chocolate bars) at the start of your proof.

11.4 A broken induction proof

What's wrong with the following induction "proof?"

Claim: all horses are the same color.

Proof: We'll show that if S is any set of horses, all horses in S have the same color, by induction on the size of S.

Base: The claim is clearly true for a set containing only one horse.

Induction: Suppose that if T is any set of $k - 1$ horses, all horses in T have the same color. Let S be a set of k horses. We need to show that all horses in S have the same color.

Suppose S contains horses $H_1, H_2, \ldots, H_k$. The set $S' = \{H_2, \ldots, H_k\}$ contains only $k-1$ horses, so they must all be the same color by the inductive hypothesis. Similarly, all the horses in the set $S'' = \{H_1, \ldots, H_{k-1}\}$ must be the same color. Since S is the union of S' and S'', all the horses in S must have the same color.

12. Recursive definition

12.1   Induction on recursive definition

For each of the following functions, compute the first few values of the function and then prove the closed form is correct.

(a) Define a function $g : \mathbb{Z}^+ \to \mathbb{Z}$ by

$$g(1) = 1$$
$$g(n) = g(n-1) + 6n - 6 \text{ (for all integers } n >= 2)$$

Closed form: $g(n) = 3n^2 - 3n + 1$

(b) Define a function $g : \mathbb{N} \to \mathbb{N}$ by

$$g(0) = 0$$
$$g(n) = n + 3g(n-1) \text{ for all integers } n \geq 1$$

Closed form: $g(n) = \frac{3^{n+1} - 2n - 3}{4}$

(c) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$$f(1) = 3, \ f(2) = 5$$
$$f(n) = 3f(n-1) - 2f(n-2) \text{ for all } n \geq 3.$$

Closed form: $f(n) = 2^n + 1$

(d) Define a sequence of values x_n as follows:

$$x_1 = 1, \ x_2 = 7$$
$$x_{n+1} = 7x_n - 12x_{n-1} \text{ for } n \geq 2$$

Closed form: $x_n = 4^n - 3^n$

12.2   Unrolling

Find closed forms for the following recursive definitions using unrolling. Specifically, show at least two steps of unrolling, a summation whose value is equal to $T(n)$, and finally a closed-form expression (i.e. containing no recursion or summations) equal to $T(n)$.

(a) $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$$T(1) = 1$$
$$T(n) = 2T(n-1) + 3$$

(b) $f : \mathbb{N}+ \to \mathbb{N}$ defined by

$$f(0) = 0$$
$$f(n) = 5f(n-1) + 1$$

(c) $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined (powers of 3 only) by

$$T(1) = 47$$
$$T(n) = 3T(n/3) + 13n$$

12.3 Graphs and recursion

Let's define a new set of graphs X_n as follows:

1. X_1 consists of two vertices and no edges.

2. For every $k \geq 2$, X_k consists of a copy of X_{k-1} plus two additional vertices. There is an edge from each of the additional vertices to each vertex in the copy of X_{k-1}.

For example, the following figure shows X_1, X_2, X_3, and X_4. A star marks the vertices in the copy of X_{k-1}.
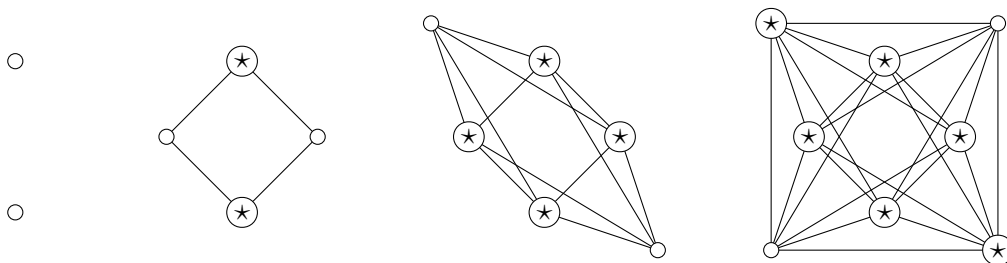
Suppose that V_k and E_k are the number of vertices and edges in X_k.

(a) Give a table showing the number of vertices and the number of edges in X_k, for k from 1 through 6.

(b) Give a formula for V_k.

(c) Write a recursive definition for E_k.

(d) Find a closed form expression E_k.

(e) The distance between two vertices a and b is the number of edges on the shortest path from a to b (which is zero if $a = b$). The diameter of a connected graph is the maximum distance between any two vertices. For $k \geq 2$, what is the diameter of X_k? Briefly explain why your answer is correct.

13.   Trees

Some useful standard closed forms:

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1} \quad (r \neq 1)$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

13.1   Recursion trees

Use recursion trees to find closed forms for the following. Ensure that the input is always an integer by assuming, as needed, that n has a special form (e.g. a power of 2).

(a) $T(1) = 47$
$T(n) = 3T(n/3) + 13n$ for $n \geq 2$

(b) $T(1) = 1$
$T(n) = 2T(n - 1) + 3$ for $n \geq 2$

(c) $T(1) = 4$
$T(n) = 2T(\frac{n}{2}) + n + 1$ for $n \geq 5$

13.2   Grammar Trees

(a) Define a grammar G_1 by $S \to aSbS \mid SaS \mid ab \mid a$. where S is the only start symbol and the terminal symbols are a and b. Prove that a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b.

(b) Define a grammar G_2 as follows, in which S is the only start symbol and the terminal symbols are a and b.

$$S \to bA \mid aB \mid SS$$
$$A \to aS \mid a$$
$$B \to bS \mid b$$

Prove that, in a tree generated by G_2, there are an equal number of a nodes and b nodes.

(c) Define a grammar G_3 by $S \to aS \mid aSS \mid a$, in which S is the only start symbol and a is the only terminal symbol. Prove that in any tree generated by G_3 there are an equal number of nodes labeled S as nodes labeled a.

13.3   Non-grammar tree inductions

(a) The Fibonacci trees T_n are a special sort of binary trees defined recursively as follows.

   (1) T_1 and T_2 are binary trees with only a single vertex.
   (2) For any $n \geq 3$, T_n consists of a root node with T_{n-1} as its left subtree and T_{n-2} as its right subtree.

Use induction on the n to prove that the height of T_n is $n - 2$, for any $n \geq 2$.

(b) A parity tree is a full binary tree with each node colored orange or blue, such that:

   (1) If v is a leaf node, then v is colored orange.
   (2) If v has two children of the same color, then v is colored blue.
   (3) If v has two children of different colors, then v is colored orange.

Prove by induction that every parity tree has the *parity property:* if the root is colored orange, then it has an odd number of leaves; and if the root is colored blue, then it has an even number of leaves.

13.4 Challenge Example

A **binomial tree** of order k is defined recursively as follows:

(1) A single root node is a binomial tree of order 0.

(2) A binomial tree of order k consists of two binomial trees of order $k-1$, with the root of the first connected as the rightmost child of the root of the second.

The following picture shows the binomial trees of order 1, 2, and 3. The labels on the nodes show how the larger tree is divided into two lower-order subtrees.
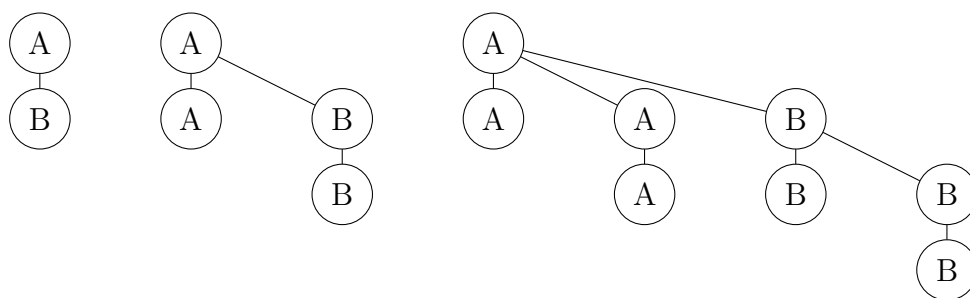
(a) Use induction on the order of the tree to prove that a binomial tree of order k has 2^k nodes.

(b) Use induction on the order of the tree to prove that a binomial tree of order k has exactly $\binom{k}{i}$ nodes at level i. Hint: For some randomly chosen level i, sum the numbers of nodes in the two trees.

Warning: in your inductive step you **must** divide the larger tree into smaller subtrees by taking it apart *at the root.* Do not try to graft things onto the bottom of a small tree to make a big one.

14. Big-O

14.1 Induction with Inequalities

(a) Prove that $n^2 > 7n + 1$ for all integers $n \geq 8$

(b) Prove that $\sum_{k=n+1}^{2n} \frac{1}{k} \geq \frac{7}{12}$ for all integers $n \geq 2$

(c) Prove that $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq \frac{3}{4} - \frac{1}{n}$ for all integers $n \geq 2$

⋆(d) Define a function $f : \mathbb{N} \to \mathbb{N}$ recursively by:

$$f(0) = 0$$
$$f(k) = k + f(\lfloor \tfrac{k}{3} \rfloor) + f(\lfloor \tfrac{k}{5} \rfloor) + f(\lfloor \tfrac{k}{7} \rfloor) \text{ for every } k > 0$$

Use (strong) induction to prove that $f(k) < 4k$ for every $k > 0$.

Hint: Carefully check each use of the inductive hypothesis to make sure it refers to an integer covered by your base case(s) or your inductive hypothesis. Notice that the claim is **not** true for zero, so you can't apply the inductive hypothesis to zero.

14.2 Big-O Analysis

For the following problems it is enough to propose values for C and k and briefly justify these by showing that they satisfy the defining inequality for big-o notation.

(a) Show that $f(n) = n^2 + 8n + 2$ is $\Theta(n^2)$.

(b) Show that $\frac{x^3 + 2x}{2x + 1}$ is $O(x^2)$.

(c) Show that $\frac{x^4 - 1}{x^2 + 1}$ is $O(x^2)$.

(d) Show that $2^x + 17$ is $O(3^x)$.

15. Algorithm Analysis

15.1 Sorting an Almost Sorted Array

The algorithm below sorts an array of integers which is "almost" sorted in the sense that every integer starts off at distance at most k from its position in the sorted (in ascending order) array. To be precise, let the position of an integer in the unsorted array be i and let the position of the integer in the array after sorting be j. Then if $|i - j| \leq k$ for all the values in the input array, the output array will be completely sorted.

The function $minimum$ returns the smaller of its two inputs. You may assume that the evaluation of $minimum(a, b)$ takes place in constant time. The function $swap$ replaces the value of a_i in the array of integers with the value of a_j and vice versa. You may assume that the evaluation of $swap(a_i, a_j)$ also requires constant time.

```
00    AlmostSorted(k,a₁, a₂, . . . , aₙ: an integer and an array of n integers)
01       for i = 1 to n
02          min=aᵢ;
03          minpos=i;
04          for j = i + 1 to minimum(i + k, n)
05             if (aⱼ < min)
06                min=aⱼ
07                minpos=j
08          end for
09          swap(aᵢ, a_minpos)
10       end for
11       return a₁, a₂, . . . , aₙ
```

Let $T(n)$ denote the running time of AlmostSorted on an input array of length n. Find a tight big-O bound on $T(n)$. Briefly justify your answer.

15.2   Mystery Code I

In line 05 the procedure maxthree is calling itself on a version of the input list with the kth element (a_k) removed. Assume it takes constant time to temporarily remove a_k from the list. (Doing this in constant time actually requires some extra details that we are hiding for clarity.)

```
00 maxthree(a_1, ..., a_n) : list of n positive integers, n ≥ 3)
01        if (n = 3) return a_1 + a_2 + a_3
02        else
03                bestval = 0
04                for k = 1 to n
05                        newval = maxthree(a_1, a_2, ..., a_{k-1}, a_{k+1}, ... a_n)
06                        if (newval > bestval) bestval = newval
07                end for
08                return bestval
```

(a) Describe (in English) what maxthree computes.

(b) Suppose that $T(n)$ is the running time of maxthree on an input array of length n. Give a recursive definition of $T(n)$.

(c) How many leaf nodes are there in the recursion tree for $T(n)$? Briefly explain.

(d) Does maxthree run in $O(2^n)$ time? Briefly explain why or why not.

15.3 Mystery Code II

The procedure crunch takes an array of integers $a_1, a_2, \ldots, a_n$ (where $n \geq 1$) and returns an integer. Assume that dividing the array in two (line 05) requires only constant time.

```
00    crunch(a_1, ..., a_n : array of integers)
01        if (n = 1 and a_1 ≥ 0) return 1
02        else if (n = 1) return 0
03        else
04            m = ⌊n/2⌋
05            output = crunch(a_1, ..., a_m) + crunch(a_{m+1}, ..., a_n)
06            return output
```

(a) Give a succinct English description of what crunch computes.

(b) Suppose that $T(n)$ is the running time of crunch on an input array of length n. Write down a recurrence for $T(n)$, including a base case. For simplicity, you may assume that n is a power of 2.

(c) What is the big-Θ running time of crunch? Justify your answer either by unrolling the recurrence or by drawing a (well-labeled) recursion tree.

15.4 Mystery Code III

Consider an array of n *distinct* real numbers $a_1, a_2, \ldots, a_n$. We say that the array has a *peak* at position k if the following two conditions hold for every position j between 2 and n:

(1) If $j \leq k$, then $a_{j-1} < a_j$.

(2) If $j > k$, $a_{j-1} > a_j$.

Consider the following procedure to determine position of the peak of an array (assume that the array does indeed have a peak):

```
00  procedure FindPeak(a₁, a₂, ..., aₙ: array of real numbers)
01      if (n = 1)
02          return 1
03      if (a₁ > a₂)
04          return 1
05      else if (aₙ > aₙ₋₁)
06          return n
07      k = floor((1+n)/2)
08      if (aₖ₋₁ > aₖ)
09          return FindPeak(a₁, ..., aₖ₋₁)
10      else if (aₖ < aₖ₊₁)
11          return FindPeak(aₖ₊₁, ..., aₙ) + k
12      else
13          return k
```

(a) Consider the array $-1, 3, 6, 7, 0$. Trace the execution of the above pseudocode and show that it correctly returns the position of the peak.

(b) At line 07, what is the smallest value that n might contain? Why?

(c) Let $T(n)$ be the worst-case running time of the above pseudocode when the array has size n. Write a recurrence for $T(n)$, including the necessary base case(s). Assume that splitting the array (lines 09 and 11) takes constant time.

(d) What is the big-Θ running time of FindPeak? Justify your answer by solving the recurrence via unrolling or use of a recursion tree.

15.5 Recursive versus Iterative Algorithms

Here is the code for a mysterious algorithm named Foo.

```
00   Foo(n: non-negative integer)
01      if n = 0 or n = 1
02         return n
03      else
04         a := 0
05         b := 1
06         for i := 2 to n
07            temp := b
08            b := b + a
09            a := temp
10         end for
11      return b
12      end if
```

(a) Give a brief English description of what the function Foo computes.

(b) What is the big-O running time of Foo? Justify your answer.

(c) A simple recursive version of Foo exists, which computes the value of Foo(n) using the values Foo(n-1) and Foo(n-2). Write the corresponding pseudocode for RecursiveFoo. Briefly explain how RecursiveFoo works.

(d) Briefly justify why Foo is more efficient than RecursiveFoo.

16.   Proof by Contradiction

(a) Prove that $\sqrt{2} + \sqrt{6} < \sqrt{15}$.

(b) Prove that there are infinitely many integers n of the form $n = 4k + 3$.

(c) (Slightly harder proof from Rosen.) Prove that there is no rational number r for which $r^3 + r + 1 = 0$.

   Hint: Assume that $r = a/b$ is a root and that a/b is in lowest terms. Multiply the equation by b^3 to clear denominators. Then consider the oddness/evenness of a and b.

17.   Collections of Sets

Recall that, for some set S, the power set of S, $\mathbb{P}(S)$, is a set of all possible subsets of S. The size of $\mathbb{P}(S)$ is $2^{|S|}$.

17.1   Power Sets 1

Define the following sets:

$$
\begin{aligned}
A &= \{68, 28\} \\
B &= \{\text{rain}, \text{snow}, \text{sun}\} \\
C &= \{\text{water}, \text{ice}\} \\
D &= \{\{\text{water}\}, \{\text{milk}\}\} \\
E &= \{(\text{water}, \text{ice})\} \\
F &= \{\text{ink}\}
\end{aligned}
$$

List the elements of each of the following sets or calculate the cardinality (as indicated).

(a) $\mathbb{P}(B)$

(b) $\mathbb{P}(E)$

(c) $\mathbb{P}(C) - D$

(d) $\mathbb{P}(C) \cap \mathbb{P}(E)$

(e) $|\mathbb{P}(A \cup B) \cup \mathbb{P}(D \cup E)|$

17.2   Power Sets 2

Define the following sets:

$$
\begin{aligned}
A &= \{\{\text{Elm}\}, \{\text{Pine}\}\} \\
B &= \{\text{Elm}, \text{Oak}, \text{Maple}\} \\
C &= \{\text{Elm}, \text{Vine}, \text{Birch}, \text{Maple}\} \\
D &= \{\text{Tree}, \text{Disease}, \text{Street}\}
\end{aligned}
$$

List the elements of each of the following sets or calculate the cardinality (as indicated).

(a) $\{X \in \mathbb{P}(C) \ : \ |X| \text{ is even}\}$

(b) $A \cap \mathbb{P}(B \cap C)$

(c) $|\mathbb{P}(C \times D)|$

(d) $|\mathbb{P}(B \cap D)|$

17.3   Set-valued Functions

Define $f : \mathbb{Z} \to \mathbb{P}(\mathbb{Z})$ by $f(n) = \mathbb{Z} - \{n\}$. Compute the following values:

(a) $f(37) \cap f(10)$

(b) $\overline{f(3) \cap f(4)}$

(c) $f(3) - (f(4) \cap f(7))$

17.4   Partitions

Recall that a partition $\mathcal{P}$ of a (finite) set S is a collection of subsets (denoted $S_1, \ldots, S_n$) of S that satisfies the following three properties:

(1) $\mathcal{P}$ covers all of S : $S_1 \cup S_2 \cup \ldots \cup S_n = S$

(2) $\mathcal{P}$ contains no empty sets: $S_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$

(3) $\mathcal{P}$ contains no overlapping sets: $S_i \cap S_j = \emptyset$ whenever $i \neq j$

Suppose that $S = \{a, b, c, d, e, f, g\}$. Determine whether each of the following sets is a partition of S. Explain why or why not.

(a) $\{\{c, b, f\}, \{a, g\}, \{e\}, \{d\}\}$

(b) $\{\{c, b, f\}, \{b, d, e\}, \{a, g\}, \emptyset\}$

(c) $\{\{c, b, f\}, \{a, g\}, \{e\}, \{d\}, \{\emptyset\}\}$

(d) $\{\{c, h, f\}, \{d, e\}, \{a, g, b\}\}$

(e) $\{\{a, b, c, d, e, f, g\}\}$

(f) $\{\{\{a, b, c, d\}\}, \{\{e, f, g\}\}\}$

17.5 Counting and Combinations

Solve the following word problems. Providing brief explanations to justify your answers. You do not need to numerically compute all of the factorials. For instance, $\frac{10!}{6!}$ is an acceptable final answer.

(a) You need to form a battle group of 11 made up of orcs, elves, and goblins. In how many ways can you choose the composition of your battle group?

(b) How many ways can you construct a string of 20 decimal digits that contains exactly 3 zeros, no two of which are consecutive? (Hint: set up the other 17 digits with spaces between them and at the ends. Pick three of these spaces to put the zero's in.)

(c) How many bit strings of length 100 have exactly 10 zeros?

(d) Your latest cheapo cell phone keyboard only includes the uppercase alphabet (26 characters total). How many 12-character strings can you type that start with ST and contain no more than three T's? (Hint: you will need to consider cases of zero T's, one T, two T's, and three T's)

(e) Suppose a set S has 10 elements, how many subsets of S have an odd number of elements?

(f) After taking a job at Initech you have 7 managers, each of whom sends you one memo per day. Initech memos come in three types: secret, internal, and public. How many different combinations of memo types could you receive in one day? (E.g. one combination would be 1 secret, 5 internal, and 1 public, which is different from the combination 2 secret, 1 internal, and 4 public.)

17.6 A Trinomial Theorem?

(a) What is the coefficient of the $x^3 y^{14} z^{10}$-term of $(x + y + z)^{27}$?

(b) What is the coefficient of the $x^a y^b z^c$-term of $(x + y + z)^{27}$?

(c) What is the coefficient of the $x^a y^b z^c$-term of $(x + y + z)^n$?

(d) Bonus question to think about for fun: Could you write out the general formula for a "trinomial theorem"? Do you see different forms in which it might be written? How many different forms are there total? This makes a good topic for conversation when you are on a date and there seems to be too much awkward silence.

18. State Diagrams

18.1 Recovering a State Diagram From a Transition Function

Henry has built a state machine for the game Crazy Six. This game has an internal counter which starts at 0. The two players take turns putting in either the number 2 or the number 3. The machine adds numbers to its counter and declares a winner when the counter reaches a (non-zero) multiple of 6.

Henry is out sick today and you are asked to complete the state diagram.

1. Draw a picture of the state diagram based on the text file given on the next page. Organize your picture and draw it clearly, so that it is easy to read. Be sure to mark the start and final states.

2. One of the sticky notes on Henry's desk says that the machine does the wrong thing on certain sequences of inputs. Find and describe the bugs that are causing the wrong behaviors. Hint: write a meaningful label for each state.

3. Another sticky note suggests that the machine seems to have too many states. Are there any pairs of states which could be combined into a single state without changing the behavior of the machine?

4. Revise the state diagram so that it has the right behavior and no unnecessary states. Change the existing state diagram as little as possible (e.g. keep the same state names).

```
Henry's Game

States: A, B, C, D, E, F, G, H, K
Actions: 2, 3, write-win
Start states: A
Final states: K


Transition function

(A,2): B
(A,3): G
(B,2): F
(B,3): E
(C,2): G
(C,3): H
(D,write-win): K
(E,2): C
(E,3): F
(F,2): D
(F,3): C
(G,2): E
(G,3): A
(H,2): D
(H,3): C
```
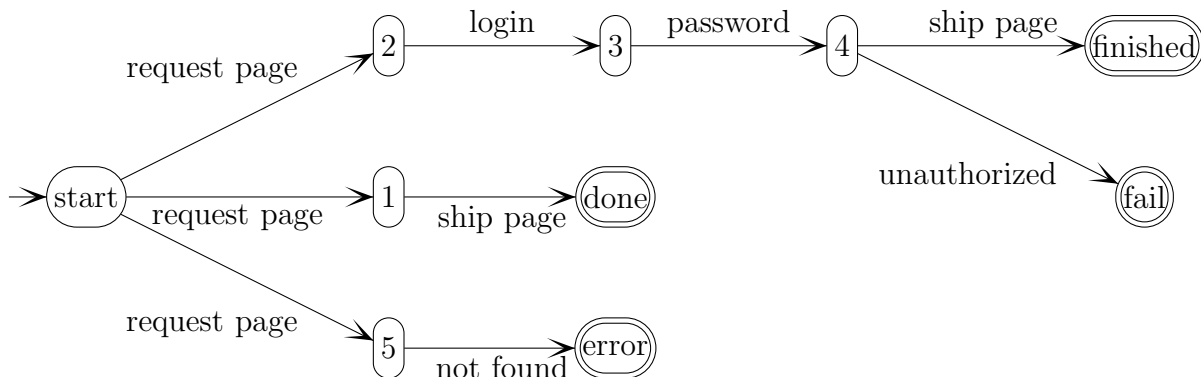
18.2   A Simple State Diagram

This state diagram represents how a user might retrieve a web page via http.

(a) Pick two states and write out the directed walk between them. Remember that the full version of the walk contains both a sequence of states and a sequence of actions.

(b) Let S denote the set of states and A denote the set of actions in the diagram. Explicitly list the contents of S and A.

(c) Recall that the transition function δ for a state diagram is a function that takes a state/action pair as input and returns the *set* of potential new states. What is the value of δ on the following inputs: (1, request page), (3, password), and (start, request page)?

(d) How large is $S \times A$? For how many of these values does δ return a value other than the empty set?

(e) In practice, this state diagram may be more complicated than it needs to be. Can you think of ways to modify the diagram to make it smaller? (For instance, we might decide that we can combine "done" and "finished" into a single state.) Note that modifications to the diagram may alter the transition function, however, you should avoid modifications which change the set of possible sequences of actions.

19. Countability

An infinite set A is called **countably infinite** if there exists a bijection $f : \mathbb{N} \to A$.

A set is called **countable** if it is either finite or countably infinite.

A set that is not countable is called **uncountably infinite** or just **uncountable**.

19.1 Which Kind of Infinity?

State whether each of the following sets is countably infinite or uncountable.

(a) $\mathbb{N}$

(b) $\mathbb{P}(\mathbb{N})$

(c) $\mathbb{C}$ (the complex numbers)

(d) $X = \{S : S \subset \mathbb{N}, S \text{ is finite}\}$

(e) The number of books written in Unicode that could ever be written

(f) Real numbers r such that r's decimal expansion eventually ends with the decimal sequence 141592653

19.2   A Curious Bijection

Recall that two sets have the same size (cardinality) if there is a bijection between them. You might imagine that $\mathbb{N}^2$ has more elements than $\mathbb{N}$. However, in this problem we will investigate a function $f : \mathbb{N}^2 \to \mathbb{N}$ that turns out to be a bijection and, therefore, shows these two sets are, in fact, the same size.

Before defining f, let us agree to denote by $s(n)$ the sum of the first n natural numbers. In other words, s is the following function:

$$s : \mathbb{N} \to \mathbb{N} \text{ by } s(n) = \sum_{i=0}^{n} i$$

Now we define f in terms of s as follows:

$$f : \mathbb{N}^2 \to \mathbb{N} \text{ by } f(x, y) = s(x + y) + x$$

The following questions will explore some properties of this seemingly complicated function:

(a) Draw a picture of what the function does for pairs (x, y) such that $x+y \le 4$. Specifically, draw the usual xy-coordinate plane, and, at the (x, y)-position, write the output value $f(x, y)$. For example, the location $(1, 2)$, should contain the value 7.

(b) For some fixed natural number k, consider all pairs (x, y) with $x + y = k$. What range of output values does f produce for this set of input pairs?

(c) What is the preimage of the output value 17?

(d) Suppose that we have two pairs (x, y) and (p, q) which are not the same and for which $x + y \ne p + q$. Explain why $f(x, y) \ne f(p, q)$.

(e) Suppose that we have pairs (x, y) and (p, q) which are not the same but for which $x + y = p + q$. Explain why $f(x, y) \ne f(p, q)$.

The answers to (d) and (e) could easily be combined into a formal proof that f is one-to-one. Similarly, the pattern observed in (a), (b), and (c) leads to a proof that f is onto.
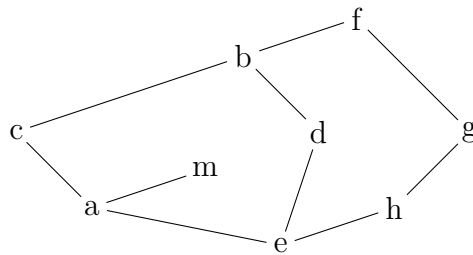
20. Planar Graphs

A graph G is a subdivision of another graph H if G can be obtained from H by adding new nodes of degree 2 on one or more of the edges of H.

Kuratowski's theorem: A graph is planar iff it does not have a subgraph that is a subdivision of K_5 or $K_{3,3}$.

20.1 Faces

For the graph G above, answer the following questions:

(a) How many faces does G have?

(b) What is the degree of each face of G?

(c) Verify the handshaking theorem for faces (the sum of the face degrees is twice the number of edges) for G.

(d) Verify that Euler's formula $(f - e + v = 2)$ holds for this graph.

20.2   Euler's Formula

Euler's formula states that $f - e + v = 2$ for a **connected planar graph**. Generalize this formula to the case where the graph might have more than one connected component. Start by drawing a graph with more than one connected component, and figure out the formula then.

   Prove your result by induction on the number of connected components. Hints: the normal version of Euler's formula is the base case. What is a small modification to a graph that would reduce the number of connected components by one?

20.3   Planar graphs

Here are two graphs X_3 (left) and X_4 (right).
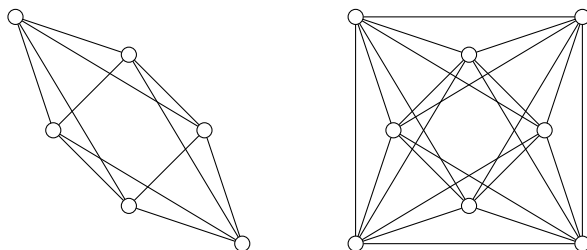
(a) Show that X_3 is planar by redrawing it so that pairs of edges never cross.

(b) A graph cannot be planar if it contains $K_{3,3}$ or K_5 as a subgraph. Show that X_4 is not planar by showing that it has a subgraph that is isomorphic to one of these two special graphs. The most effective way to do this is probably a combination of a labelled picture (start with what you did for part a) combined with some explanation.

(c) Suppose that G is an undirected connected simple planar graph with 8 vertices, 2 of which have degree 4 and 6 of which have degree 3. How many edges does it have? How many faces does it have? (Hint: use Euler's formula.)