Nested Quantifiers

Explain why each of the following propositions is true or false:

(a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}, y^3 \leq x$

(b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y^3 \leq x$

(c) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N}, \mathrm{GCD}(x, y) = 1$

(d) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z}, x = y^2$

Solution:

(a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}, y^3 \leq x$
TRUE

(b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y^3 \leq x$
FALSE For any choice of y there is a smaller x

(c) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N}, \mathrm{GCD}(x, y) = 1$
TRUE $x = 1$ will give $\mathrm{GCD}(x, y) = 1$ for all y

(d) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z}, x = y^2$
FALSE $x = 2$ there is no y that when squared is 2 in the integers

Function Properties Determine whether each function is onto and/or one-to-one. Briefly explain why it is or give a concrete counter-example showing why it is not. Warning: check each definition to make sure the function is properly defined, e.g. exactly one output for each input, output values all lie in declared co-domain.

(a) $b : \mathbb{Z} \to \mathbb{Z}$ by $b(n) = 2^n$

(b) $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3 + 7$

(c) $l : \mathbb{R} \to \mathbb{R}$ by $l(x) = \lfloor x \rfloor$

(d) $m : \mathbb{N}^2 \to \mathbb{N}$ by $m(x, y) = x - y$

(e) $p : \mathbb{Z}^2 \to \mathbb{Z}$ by $p(x, y) = xy$

Solution:

(a) $b : \mathbb{Z} \to \mathbb{Z}$ by $b(n) = 2^n$
This is not a function since if $n < 0$ the result will not be an integer

(b) $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3 + 7$
This is both one-to-one and onto

(c) $l : \mathbb{R} \to \mathbb{R}$ by $l(x) = \lfloor x \rfloor$
This is a function but neither one-to-one nor onto

(d) $m : \mathbb{N}^2 \to \mathbb{N}$ by $m(x, y) = x - y$
This is not a function since in all cases where $y > x$ the result is not a natural number

(e) $p : \mathbb{Z}^2 \to \mathbb{Z}$ by $p(x, y) = xy$
This is onto but not one-to-one

One-to-One Prove that the following functions is one-to-one.

$$\mathbb{N} \rightarrow \mathbb{Z} \text{ by } h(x) = x^2 + 42$$

Solution:

Let $a, b \in \mathbb{N}$ such that $h(a) = h(b)$
$a^2 + 42 = b^2 + 42$
$a^2 = b^2$
Since $a, b \in \mathbb{N}$, $\sqrt{a^2} = a$ and $\sqrt{b^2} = b$
So $a = b$
Thus $\forall a, b \in \mathbb{N}$ if $h(a) = h(b)$ then $a = b$ which is the definition of one-to-one so h is one-to-one. $\square$

Onto Prove that the following function is onto

$$\mathbb{Z}^2 \to \mathbb{Z} \text{ by } f(x, y) = xy + 27$$

Solution:

Let $c \in \mathbb{Z}$
Consider $a = 1$ since 1 is an integer a is clearly an integer.
Consider $b = c - 27$ since c and -27 are integers clearly b is an integer.
So $(a, b) \in \mathbb{Z}^2$
$f(a, b) = f(1, c - 27) = 1(c - 27) + 27 = c - 27 + 27 = c$
So for any $c \in \mathbb{Z}$, (a, c) is the preimage of c for f which is the definition of onto so f is onto. $\square$