Power Sets
Define the following sets:

$$\begin{aligned}A &= \{68, 28\}\\ B &= \{\text{rain}, \text{snow}, \text{sun}\}\\ C &= \{\text{water}, \text{ice}\}\\ D &= \{\{\text{water}\}, \{\text{milk}\}\}\\ E &= \{(\text{water}, \text{ice})\}\\ F &= \{\text{ink}\}\end{aligned}$$

List the elements of each of the following sets or calculate the cardinality (as indicated).

(a) $\mathbb{P}(B)$

(b) $\mathbb{P}(E)$

(c) $\mathbb{P}(C) - D$

(d) $\mathbb{P}(C) \cap \mathbb{P}(E)$

(e) $|\mathbb{P}(A \cup B) \cup \mathbb{P}(D \cup E)|$

Solution:

(a) $\mathbb{P}(B) = \{\emptyset, \{\text{rain}\}, \{\text{snow}\}, \{\text{sun}\}, \{\text{rain}, \text{snow}\}, \{\text{rain}, \text{sun}\}, \{\text{sun}, \text{snow}\}, \}, \{\text{rain}, \text{snow}, \text{sun}\}\}$

(b) $\mathbb{P}(E) = \{\emptyset, \{(\text{water}, \text{ice})\}\}$

(c) $\mathbb{P}(C) - D = \{\emptyset, \{\text{ice}\}, \{\text{water}, \text{ice}\}\}$

(d) $\mathbb{P}(C) \cap \mathbb{P}(E) = \{\emptyset\}$

(e) $|\mathbb{P}(A \cup B) \cup \mathbb{P}(D \cup E)| = 1 + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} + 1 + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} - 1$

Partitions

Recall that a partition $\mathcal{P}$ of a (finite) set S is a collection of subsets (denoted $S_1, \ldots, S_n$) of S that satisfies the following three properties:

(1) $\mathcal{P}$ covers all of $S : S_1 \cup S_2 \cup \ldots \cup S_n = S$

(2) $\mathcal{P}$ contains no empty sets: $S_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$

(3) $\mathcal{P}$ contains no overlapping sets: $S_i \cap S_j = \emptyset$ whenever $i \neq j$

Suppose that $S = \{a, b, c, d, e, f, g\}$. Determine whether each of the following sets is a partition of S. Explain why or why not.

(a) $\{\{c, b, f\}, \{a, g\}, \{e\}, \{d\}\}$

(b) $\{\{c, b, f\}, \{b, d, e\}, \{a, g\}, \emptyset\}$

(c) $\{\{c, b, f\}, \{a, g\}, \{e\}, \{d\}, \{\emptyset\}\}$

(d) $\{\{c, h, f\}, \{d, e\}, \{a, g, b\}\}$

(e) $\{\{a, b, c, d, e, f, g\}\}$

(f) $\{\{\{a, b, c, d\}\}, \{\{e, f, g\}\}\}$

Solution:

(a) $\{\{c, b, f\}, \{a, g\}, \{e\}, \{d\}\}$ a partition

(b) $\{\{c, b, f\}, \{b, d, e\}, \{a, g\}, \emptyset\}$ not a partition

(c) $\{\{c, b, f\}, \{a, g\}, \{e\}, \{d\}, \{\emptyset\}\}$ not a partition

(d) $\{\{c, h, f\}, \{d, e\}, \{a, g, b\}\}$ not a partition

(e) $\{\{a, b, c, d, e, f, g\}\}$ a partition

(f) $\{\{\{a, b, c, d\}\}, \{\{e, f, g\}\}\}$ not a partition

Counting One

You need to form a battle group of 11 made up of orcs, elves, and goblins. In how many ways can you choose the composition of your battle group?

Solution:

What you have here is 3 types with 11 slots. Using the formula you get:

$$\binom{11+3-1}{3-1} = \binom{13}{2}$$

The idea is that you are building a string to represent the orcs, elves, and goblins as 11 x's and two splits. You can make all the combinations of strings by selecting either the places for the two splits or the 11 x's.

$$\binom{13}{2} = \binom{13}{11}$$

Counting Two

You need to form a battle group consisting of an orc, an elf, and a goblin whose total strength is 12. The strength of each creature is an integer between 1 and 10 and the strength of the group is the sum of the individual strengths. In how many ways can you construct your battle group?

Solution:

There are two key ideas here the first is that you can treat the total strength 12 items that you split between three categories. That would seem to give you

$$\binom{12+3-1}{3-1} = \binom{14}{2}$$

The problem is that would allow some of the battle group to have 0 strength or 12 strength but they can only have 1-10. So one way to fix this is give each member 1 strength and then distribute the remaining 9 between them which gives.

$$\binom{9+3-1}{3-1} = \binom{11}{2}$$