Inclusion proof Claim: If $A = \{(x, 5 - (x-3)^2) \mid x \in [1,5]\}$ and $B = \{(x,y) \mid x, y \in \mathbb{R}, x \geq 0, y \geq 0\}$ then $A \subset B$.

Definition: A is a subset of B if every element of A is also an element of B.

Proof:

Let $A = \{(x, 5 - (x-3)^2) \mid x \in [1,5]\}$ and $B = \{(x,y) \mid x, y \in \mathbb{R}, x \geq 0, y \geq 0\}$

Let a be an arbitrary element of A as follows $a = (p, q)$ where $p, q \in \mathbb{R}$.

Now to show a is in B we need to show p and $q \geq 0$.

We can find the zeros of $5 - (x-3)^2$, we get $x = \pm\sqrt{5} + 3$.

To show $q \geq 0$ we must show $-\sqrt{5} + 3 \leq p \leq \sqrt{5} + 3$ and some $q \geq 0$.

Since $p \in [1,5]$ and $1 > -\sqrt{5} + 3$, $5 < \sqrt{5} + 3$, and if $p = 1, q = 5 - (1-3)^2 = 1$ which is greater then 0, $q \geq 0$ for $p \in [1,5]$.

Since $p \in [1,5], q \geq 0$ and $p \in [1,5], p \geq 0$ therefor $A \subset B$. by definition of subset. $\square$

Abstract Set Proof

Claim: For any sets A and B, if $(A - B) \cup (B - A) = A \cup B$, then $A \cap B = \emptyset$

Proof:

To prove the claim we will prove the contrapositive which is.

If $A \cap B \neq \emptyset$, then $(A - B) \cup (B - A) \neq A \cup B$.

Let x be an arbitrary element in A and B. This exists since $A \cap B \neq \emptyset$.

Then $x \notin (A - B)$ since $x \in B$ and definition of difference. Also, $x \notin (B - A)$ since $x \in A$ and definition of difference.

But $x \in A \cup B$ because $x \in A$ and $x \in B$. So $x \in A \cup B$ and $x \notin (A - B) \cup (B - A)$ therefore $x \notin (A - B) \cup (B - A) \neq A \cup B$. $\square$

Extra Question

Consider the following sets.

$A = \{(a, b) \in \mathbb{Z}^2 \mid b \geq a\}$

$B = \{(x, y) \in \mathbb{Z}^2 \mid y = x^{x^2 - x}\}$

What is the relationship between A and B?

$B \subset A$

Proof:

Let $A = \{(a, b) \in \mathbb{Z}^2 \mid b \geq a\}$ and $B = \{(x, y) \in \mathbb{Z}^2 \mid y = x^{x^2 - x}\}$

Let $b \in B$ such that $b = (x, y)$ where $x, y \in \mathbb{Z}$.

Since $b \in B$ we see that $y = x^{x^2 - x}$.

Consider the exponent, $x^2 - x = x(x-1)$ since $x \in \mathbb{Z}$ it is clear that either x is even or $x - 1$ is even thus $x^2 - x$ is even. To show that $x^2 - x$ is non-negative there are four cases.

Case 1: $x = 0$ thus $0^2 - 0 = 0$ which is non-negative.
Case 2: $x = 1$ thus $1^2 - 1 = 0$ which is non-negative.
Case 3: $x > 1$ then $x^2 \geq x$ so $x^2 - x \geq 0$ which is non-negative.
Case 4: $x < 0$ then x^2 is non-negative and $-x$ is non-negative so $x^2 - x$ is non-negative.
In all cases $x^2 - x$ is non-negative.

We now want to show $x^{x^2-x} \geq x$. Here we have three cases.

Case 1: $x \leq 0$ Since anything raised to an even power is non-negative it is clear that x^{x^2-x} is non-negative since $x^2 - x$ is non-negative. So $x \leq 0 \leq x^{x^2-x}$.

Case 2: $x = 1$ In this case we get $x^{x^2-x} = 1^{1^2-1} = 1^0 = 1$ so in this case $x = x^{x^2-x} = 1$.

Case 3: $x > 1$ In this case we can tell that since x is positive and $x^2 > x$ we can see that in this case $x < x^{x^2-x}$.

In all case $y = x^{x^2-x} \geq x$. Which means that $b \in A$ $\square$