Simple Induction Prove

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Solution:

Base case $n = 0$

$\sum\limits_{i=0}^{0} 2^i = 2^0 = 1$

$2^{0+1} - 1 = 2^1 - 1 = 1$

The formula works for $n = 0$

Inductive Hypothesis:

Assume that $\sum\limits_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n < k$.

Consider $n = k$

$\sum\limits_{i=0}^{k} 2^i = 2^k + \sum\limits_{i=0}^{k-1}$ by definition.

By inductive hypothesis we get

$2^k + \sum\limits_{i=0}^{k-1} = 2^k + 2^{k-1+1} - 1$

$= 2^k + 2^k - 1 = 2 \times 2^k - 1 = 2^{k+1} - 1$

Which is what we wanted to show. So with the inductive step and base cases we have proved that $\sum\limits_{i=0}^{n} 2^i = 2^{n+1} - 1$. $\square$

Diagonal Robot A robot is walking around on the 2D integer grid. It starts at $(1, 1)$, and at each step it moves to one of the closest diagonal grid points - e.g. its first step can take it to any of $(2, 2)$, $(2, 0)$, $(0, 0)$, or $(0, 2)$. Prove that the robot can never reach the point $(0, 1)$.

Solution:

To prove that the Diagonal Robot can not reach the point $(0, 1)$ we will start by proving that all points that the Diagonal Robot can reach have a even sum. That is that for any point (x, y) that is reachable by the robot $x + y$ is even.

We will prove this by induction on the number of steps s taken by the robot.

Base Case: $s = 0$

The Diagonal Robot starts at $(1, 1)$ so the sum is 2 which is even.

Inductive Hypothesis:

Assume that for all steps $s < k$ the Diagonal Robot is only on points with an even sum.

After k steps the Diagonal Robot must be at some point $(x, y) \in \mathbb{Z}$. To have reached this point in k steps it must have been at one of the following four points in $k - 1$ steps $(x - 1, y - 1), (x - 1, y + 1), (x + 1, y - 1), (x + 1, y + 1)$ since the cover every possible move it could have taken. We will now show each of these cases.

Case $(x-1, y-1)$ Since this point was reached in $k-1$ steps by the inductive hypothesis it is even so $x - 1 + y - 1 = 2m$ for some $m \in \mathbb{Z}$. With simple algebra we get that $x + y = 2m - 2 = 2(m - 1)$ and since $(m - 1) \in \mathbb{Z}$ we can tell that $x + y$ is also even.

Case $(x+1, y-1)$ Since this point was reached in $k-1$ steps by the inductive hypothesis it is even so $x+1+y-1 = 2m$ for some $m \in \mathbb{Z}$. With simple algebra we get that $x+y = 2m$ so we can tell that $x + y$ is also even.

Case $(x-1, y+1)$ Since this point was reached in $k-1$ steps by the inductive hypothesis it is even so $x-1+y+1 = 2m$ for some $m \in \mathbb{Z}$. With simple algebra we get that $x+y = 2m$ so we can tell that $x + y$ is also even.

Case $(x+1, y+1)$ Since this point was reached in $k-1$ steps by the inductive hypothesis it is even so $x + 1 + y + 1 = 2m$ for some $m \in \mathbb{Z}$. With simple algebra we get that $x + y = 2m + 2 = 2(m + 1)$ and since $(m + 1) \in \mathbb{Z}$ we can tell that $x + y$ is also even.

Having show in all cases that after k steps $x + y$ is even we have shown with the base case that all points visited after any number of steps by the Diagonal Robot have an even sum.

The point $(0, 1)$ does not have an even sum so by the lemma we just proved is not a point that can be visited by the Diagonal Robot. $\square$