Grammar Trees

Define a grammar G_1 by $S \rightarrow aSbS \mid SaS \mid ab \mid a$. where S is the only start symbol and the terminal symbols are a and b. Prove that a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b.

Solution:

Proof by induction on the height h of the tree.

Base Cases: The grammar G_1 has no trees with height 0 but there are two trees with height 1. Those trees are generated by the rules $S \rightarrow ab$ and $S \rightarrow a$. These trees both clearly have more a nodes than b nodes so the base cases hold.

IH: Assume that for all trees defined by the grammar G_1 with height h k have at least as many a nodes as b nodes.

Now consider a Tree T defined by the grammar G_1 with height $h = k$.

There are two cases to consider that are not covered by the base cases.

Case 1: A tree with the first expansion from the root being the rule $S \rightarrow aSbS$. In this case the root will have two children that are leaves one with a label a and the other b. It will also have two children that are sub-trees T_1, T_2 each of which have S nodes as their roots so are trees of the grammar G_1 and since they are below the root of the main tree have heights $< k$. Thus by the inductive hypothesis have at least as many nodes labeled a than labeled b. So a groups of nodes have at least as many a nodes as b nodes. Thus in this case a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b

Case 2: A tree with the first expansion from the root being the rule $S \rightarrow SaS$. In this case the root will have one child that is a leaf with a label a. It will also have two children that are sub-trees T_1, T_2 each of which have S nodes as their roots so are trees of the grammar G_1 and since they are below the root of the main tree have heights $< k$. Thus by the inductive hypothesis have at least as many nodes labeled a than labeled b. So a groups of nodes have at least as many a nodes as b nodes. Thus in this case a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b

So in all cases a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b. $\square$

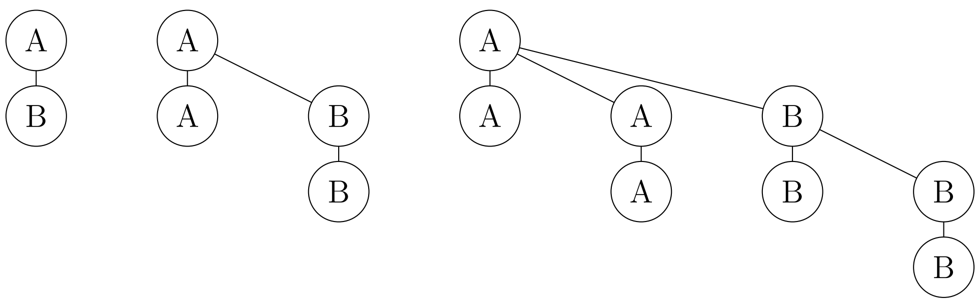
Binomial Trees

A **binomial tree** of order m is defined recursively as follows:

(1) A single root node is a binomial tree of order 0.

(2) A binomial tree of order m consists of two binomial trees of order $m-1$, with the root of the second connected as the rightmost child of the root of the first.

The following picture shows the binomial trees of order 1, 2, and 3. The labels on the nodes show how the larger tree is divided into two lower-order subtrees.



(a) Use induction on the order of the tree to prove that a binomial tree of order m has 2^m nodes.

Solution:

Proof by induction on the order m of a binomial tree.

Base Case: A binomial tree of order $m = 0$ has by definition 1 node and $2^0 = 1$.

IH: Assume that a binomial tree of order $m < k$ has 2^m nodes.

Consider a binomial tree of order $m = k$. By definition this tree is composed of two binomial trees of order $k-1$. Since $k-1 < k$ by the inductive hypothesis these each have 2^{k-1} nodes. So the total tree has $2 \cdot 2^{k-1} = 2^k$ nodes. Which shows that a tree of order k has 2^k nodes. So with the base cases and induction we have proven that all binomial trees of order m have 2^m nodes. $\square$

(b) Use induction on the order of the tree to prove that a binomial tree of order m has exactly $\binom{m}{i}$ nodes at level i. Hint: For some randomly chosen level i, sum the numbers of nodes in the two trees.

Remember

$$\binom{k}{i} = \frac{k!}{i!(k-i)!}$$

Solution:

Proof by induction on the order m of a binomial tree.
Base Case: a binomial tree of order $m = 0$ has one node and $\binom{0}{0} = 1$.
IH: Assume that a binomial tree of order $m < k$ has exactly $\binom{m}{i}$ nodes at level i.
Consider a binomial tree of order $m = k$.
To show that the number of nodes at level i is equal to $\binom{k}{i}$ there are three cases to consider.

Case 1: $i = 0$ In this case the only node at level 0 is the root of the binomial tree and $\binom{k}{0} = 1$ so in this case holds.

Case 2: $i = k$ Since a binomial tree of order k is made of two binomial trees T_1, T_2 of order $k - 1$ the nodes at level i are counted as follows. In T_1 there are no nodes at level i. This can be told since the T_1 meets the requirements of the inductive hypothesis and there is no value for $\binom{k-1}{k}$. In the case of T_2 which has its root as the rightmost child of the root of T_1 level k of the whole tree will be level $k - 1$ of the T_2 so by the inductive hypothesis it has $\binom{k-1}{k-1} = 1$ nodes at that level. Which matches the $\binom{k}{k} = 1$ of the whole tree so this case holds.

(continued)

Case 3: $0 < i < k$ Since a binomial tree of order k is made of two binomial trees T_1, T_2 of order $k-1$ the nodes at level i are counted as follows. In the case of the tree T_1 by definition and the inductive hypothesis it contributes $\binom{k-1}{i}$ nodes to level i. In the case of T_2 which is a sub-tree of the root of T_1 with the inductive hypothesis it contributes $\binom{k-1}{i-1}$ nodes to level i of the whole tree since it is lower in the tree by one level. From this we get a total number of nodes in the tree as follows.

$$\begin{aligned}
\binom{k-1}{i} + \binom{k-1}{i-1} &= \frac{(k-1)!}{i!((k-1)-i)!} + \frac{(k-1)!}{(i-1)!((k-1)-(i-1))!} \\
&= \frac{(k-1)!}{i(i-1)!(k-i-1)!} + \frac{(k-1)!}{(i-1)!(k-i)!} \\
&= \frac{(k-1)!}{i(i-1)!(k-i-1)!} + \frac{(k-1)!}{(i-1)!(k-i)(k-i-1)!} \\
&= \frac{(k-i)(k-1)!}{i(i-1)!(k-i)(k-i-1)!} + \frac{i(k-1)!}{i(i-1)!(k-i)(k-i-1)!} \\
&= \frac{(k-i)(k-1)! + i(k-1)!}{i(i-1)!(k-i)(k-i-1)!} \\
&= \frac{(k-i)(k-1)! + i(k-1)!}{i!(k-i)!} \\
&= \frac{(k-i+i)(k-1)!}{i!(k-i)!} \\
&= \frac{k(k-1)!}{i!(k-i)!} \\
&= \frac{k!}{i!(k-i)!}
\end{aligned}$$

Which shows that for all $0 < i < k$ the number of nodes at level i is $\binom{k}{i}$.

We have now shown that in all three cases the number of nodes in a binomial tree of order m at level i is $\binom{k}{i}$. $\square$