Contradiction 1 Prove that there is no rational number r for which $r^3 + r + 1 = 0$.

Solution:

Suppose for purposes of contradiction that there is a rational number r such that $r^3 + r + 1 = 0$. Since r is a rational number there are $a, b \in \mathbb{Z}, b \neq 0$ such that $r = a/b$. In this case we will choose the smallest terms a, b that meet this criteria. Thus a and b share no common divisors. Writing the equation in terms of a/b we get the following.

$$\left(\frac{a}{b}\right)^3 + \frac{a}{b} + 1 = 0$$

To simplify this we multiply through by b^3

$$a^3 + ab^2 + b^3 = 0$$

At this point we have three cases.

Case 1: a and b are odd. In this case clearly all three terms are odd so the total is odd but 0 is even so there is a contradiction.

Case 2: a and b are even. If a and b are both even then they share a factor of 2 and are not in least terms.

Case 3: a is odd b is even. In this case a^3 is even and ab^2 is even but b^3 is odd. So the sum of the three terms are odd but 0 is even so there is a contradiction.

Case 4: b is odd a is even. In this case b^3 is even and ab^2 is even but a^3 is odd. So the sum of the three terms are odd but 0 is even so there is a contradiction.

So in all four cases there is a contradiction so our supposition can't be true and thus there is no rational number r for which $r^3 + r + 1 = 0$. $\square$

Contradiction 2 Use proof by contradiction to show that $\log_5 2$ is irrational.

Solution:

Suppose not. That is, suppose that $\log_5 2$ is rational. Then $\log_5 2 = \frac{a}{b}$, where a and b are integers, b non-zero.

Raising 5 to the power of both sides, we get $2 = 5^{\frac{a}{b}}$. Raising both sides to the bth power, we get $2^b = 5^a$. Since 2 and 5 are both prime, this equation can hold only if $a = b = 0$. But we know that b is non-zero. So we have a contradiction.

Since its negation led to a contradiction, our original claim must have been true.