Sets and Modular Arithmetic Tutorial Problems

1. Congruence classes of perfect squares
   a) Compute \( \{ [x^2]_4 \mid x \in \mathbb{Z} \} \). (That is, rewrite the set into a simpler form that lists all the elements explicitly.)
   
   b) Notice that, for any \( k \), \([a]_k \neq [b]_k \) implies \( a \neq b \). (Do you see why this is true?) Using this fact and the result from part (a), prove that for all integers \( x \) and \( y \), \( x^2 + y^2 \neq 4000003 \). (Do not use a calculator.)

2. Sets warmup
   Consider the following sets: \( A = \{ 2 \} \), \( B = \{ A, \{ 4, 5 \} \} \), \( C = B \cup \emptyset \), \( D = B \cup \{ \emptyset \} \).
   
   a) Which of the sets have more than two elements?
   
   b) Which of the following are true:
      \( \begin{align*}
      2 & \in A, 2 \in B, \{ 2 \} \in A, \{ 2 \} \in B, \emptyset \in C, \emptyset \in D, \\
      \emptyset & \subseteq A, \{ 2 \} \subseteq A, \{ 2 \} \subseteq B
      \end{align*} \)

3. Cross product
   a) Find an example of sets \( A \) and \( B \) such that \( A \times B = B \times A \). Then find a second such pair of sets; try to make this second example feel different from your first, e.g. don’t just rename some elements.
   
   b) Consider the following incomplete statement:
      
      For sets \( A \) and \( B \), if \( \underline{\text{______________}} \) then \( A \times B \neq B \times A \).
      
      Create a true claim by filling in the blank with a statement about \( A \) and \( B \) that does not mention Cartesian products. Try to make the strongest possible claim, i.e. ideally your statement should still be true even if we replaced the “if-then” by an “if and only if”. If you have extra time, also prove your claim. Hint: two sets are not-equal if and only if there exists an element that is in one but not the other.