Problem 1. Consider a set $S$ defined recursively: $1 \in S$, and if $x \in S$ then $-x \in S$ and $2x \in S$.

a) What is the set $S$? (Give an equivalent non-recursive description, e.g. in set-builder notation or using sets we’ve named in the past.)

b) A recursive definition for a set is called *ambiguous* if there is any element in the set that can be constructed from the given base cases and constructors in more than one way. \(^1\) Prove the above definition for $S$ is ambiguous.

c) Is it possible to define a recursive function $f : S \rightarrow \mathbb{N}$ which counts how many constructor steps were used to produce a given element of $S$? Why or why not? (For example, we might want $f(1) = 0$ because 1 is in the base case so it looks like it doesn’t use any applications of the constructor step.)

d) Come up with a new *unambiguous* recursive definition for the same set $S$. \(^2\)

Problem 2. Consider a set $S$ defined recursively: $-2 \in S$, and if $x \in S$ then $2x \in S$ and $x^2 - 1 \in S$. You may use without proof that every element of $S$ is an integer.

a) Prove $5 \notin S$.

b) Prove $\forall y \in S(|y| \geq 2)$ by structural induction.

c) Prove that for every $y \in S$, if $y < 0$ then $y$ is even. \(^3\)

Problem 3.

a) Give a recursive definition for a function $\text{rev} : \{0, 1\}^* \rightarrow \{0, 1\}^*$ which reverses a binary string. For example, $\text{rev}(001111) = 111100$.

b) Prove by structural induction that $\text{rev}(st) = \text{rev}(t)\text{rev}(s)$.

Problem 4. Prove by structural induction: If $T$ is a binary tree, then for any labeling of $T$’s vertices using blue and orange such that the root is blue and every leaf is orange, there exists some blue vertex that has an orange child.