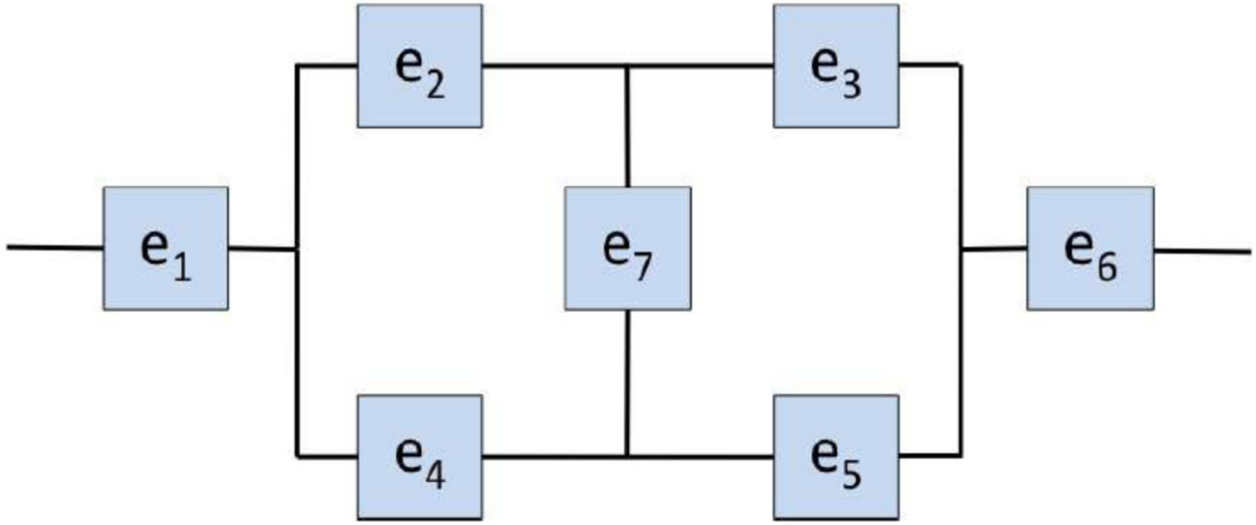
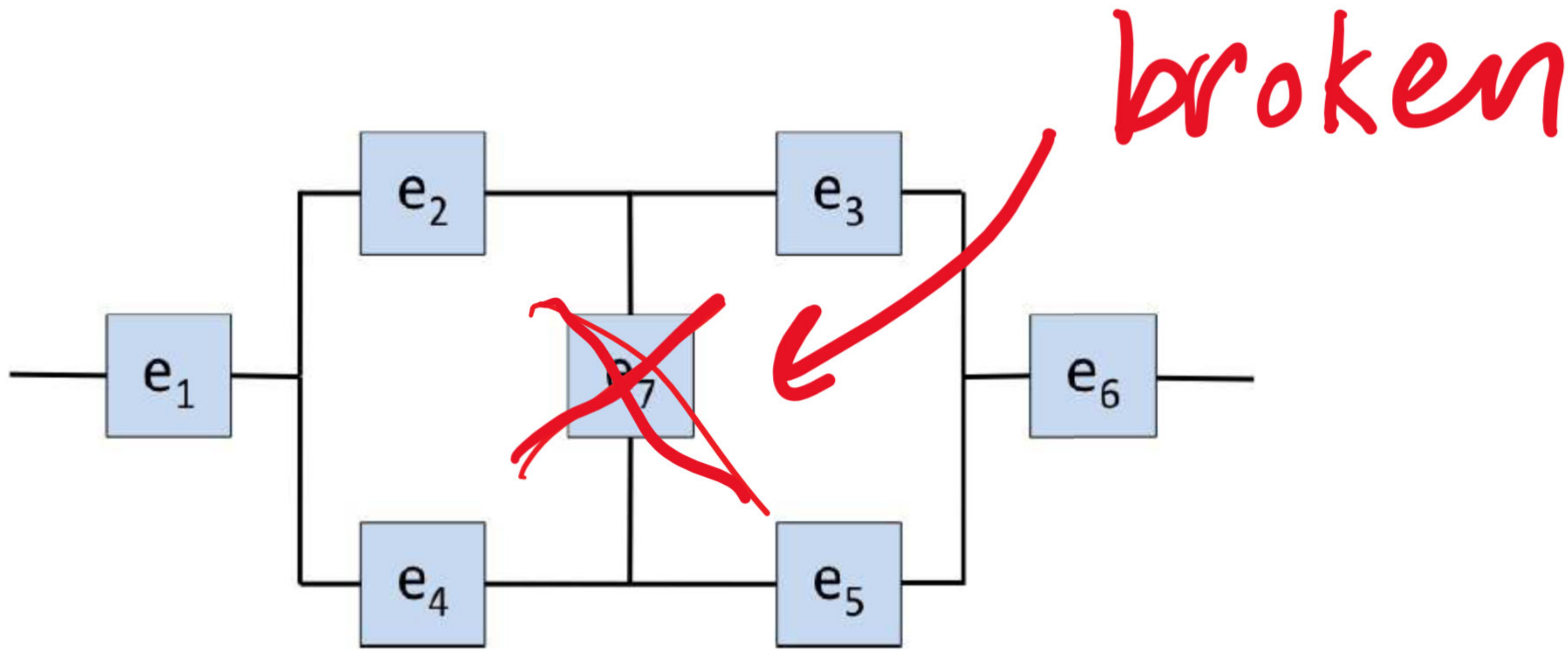


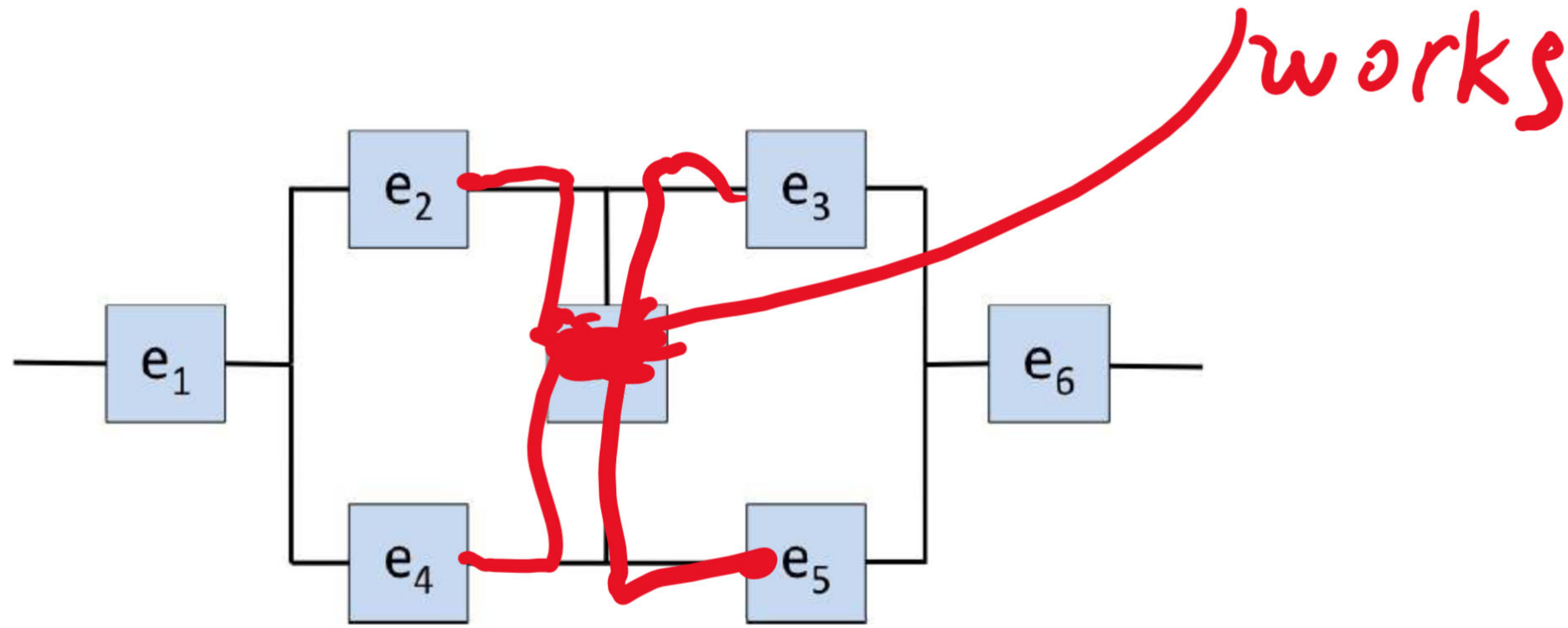
Component	e_1	e_2	e_3	e_4	e_5	e_6	e_7
Probability of component working	0.3	0.8	0.2	0.2	0.5	0.6	0.4



Component	e_1	e_2	e_3	e_4	e_5	e_6	e_7
Probability of component working	0.3	0.8	0.2	0.2	0.5	0.6	0.4

P(circuit works |e7 is broken)=P(e1 works)*
[1-(1-P(e2 works)*P(e3 works))*(1-P(e4 works)*P(e5 works))]*
P(e6 works)=0.3*(1-(1-0.8*0.2)*(1-0.2*0.5))*0.6=0.0439

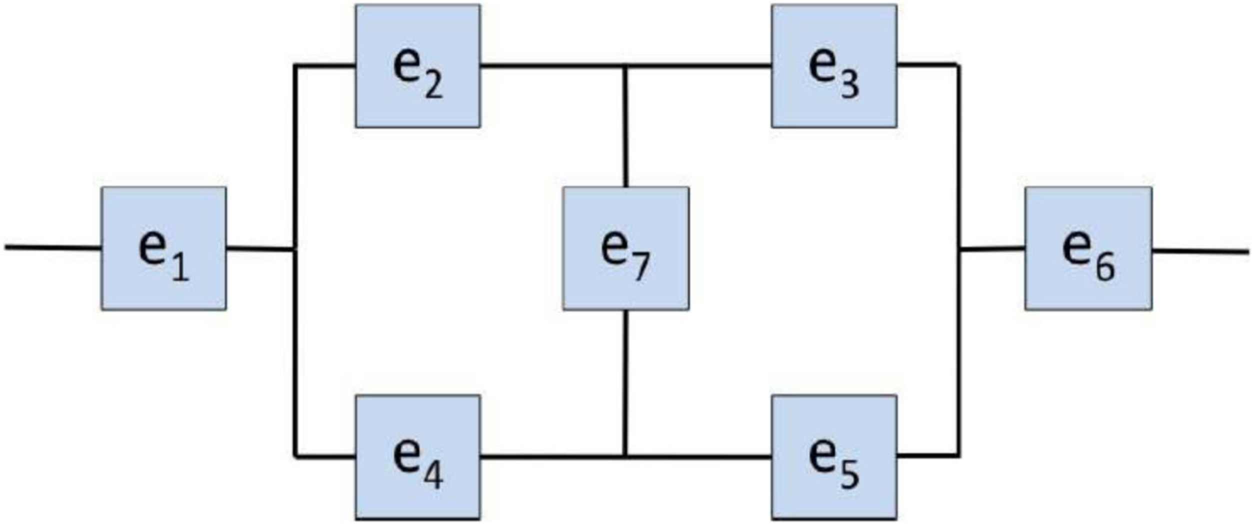
The contribution to total probability:
P(circuit works |e7 is broken)*P(e7 is broken)=0.6*0.0439=0.0264



Component	e_1	e_2	e_3	e_4	e_5	e_6	e_7
Probability of component working	0.3	0.8	0.2	0.2	0.5	0.6	0.4

P(circuit works |e7 works)=P(e1 works)*
[1-(1-P(e2 works))*(1-P(e3 works))]
[1-(1-P(e4 works))(1-P(e5 works))]*
P(e6 works)=0.3*(1-(1-0.8)*(1-0.2))*(1-(1-0.2)*(1-0.5)))*0.6=0.0907

The contribution to total probability:
P(circuit works |e7 works)*P(e7 works)=0.4*0.0907=0.0363

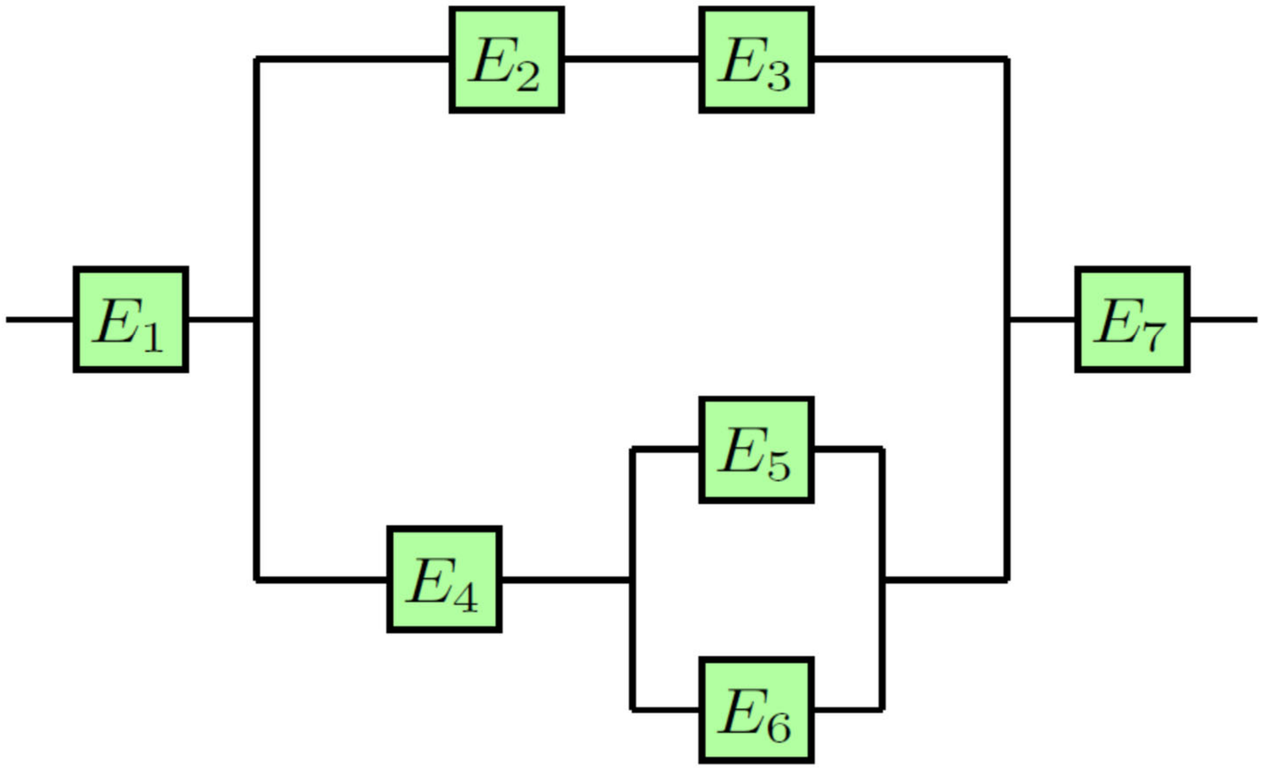


Component	e_1	e_2	e_3	e_4	e_5	e_6	e_7
Probability of component working	0.3	0.8	0.2	0.2	0.5	0.6	0.4

P(circuit works)=
P(circuit works |e7 works)*P(e7 works)+
P(circuit works |e7 is broken)*P(e7 is broken)=
=0.0264+0.0363=0.0627

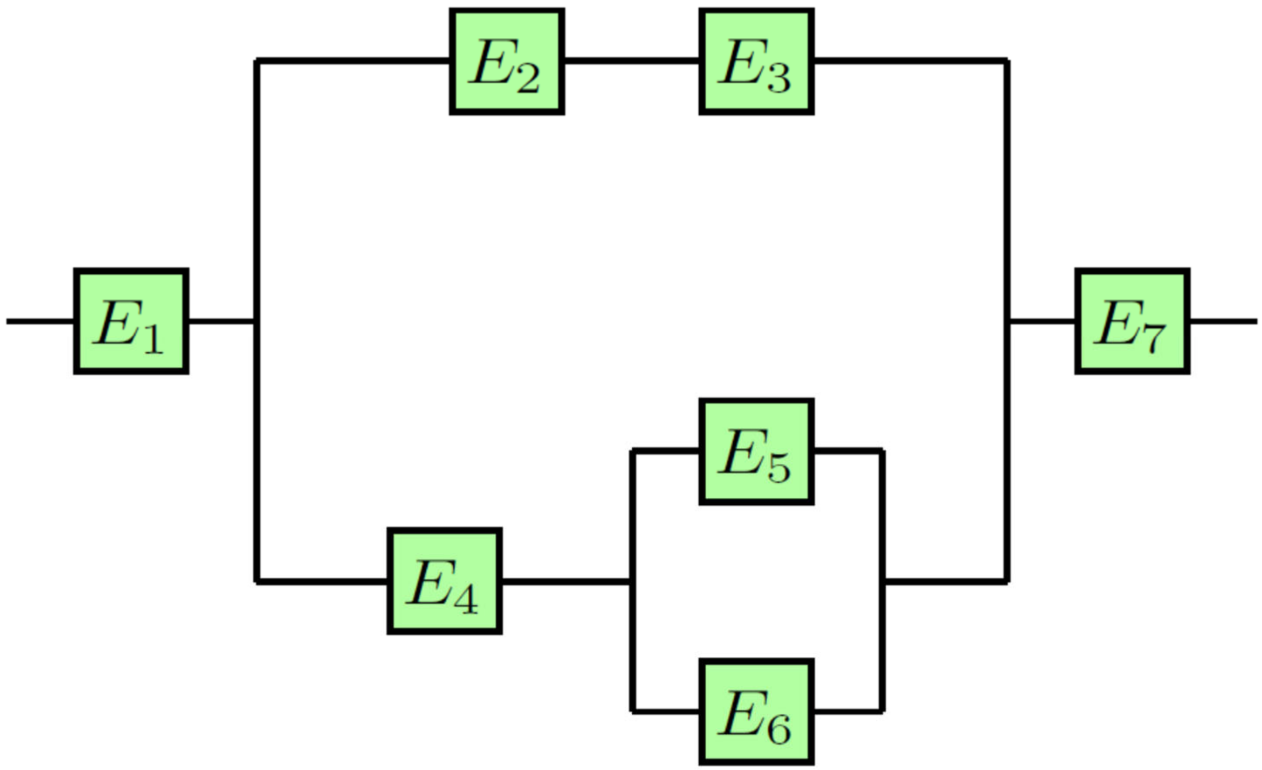
Answer: 6.27%

Circuit → Set equation



Component	E_1	E_2	E_3	E_4	E_5	E_6	E_7
Probability of functioning well	0.9	0.5	0.3	0.1	0.4	0.5	0.8

Circuit → Set equation

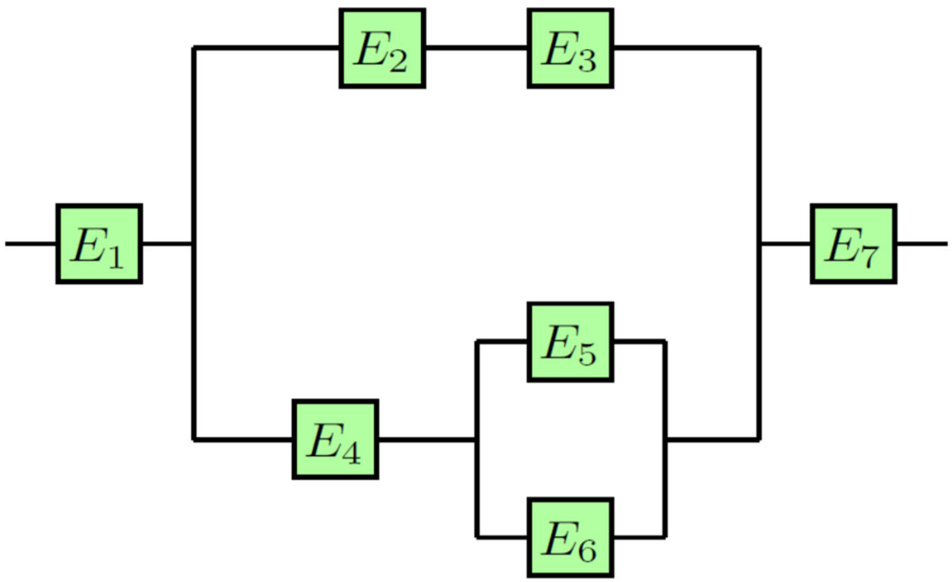


Component	E_1	E_2	E_3	E_4	E_5	E_6	E_7
Probability of functioning well	0.9	0.5	0.3	0.1	0.4	0.5	0.8

P(Works) = 0.9.*(1-(1-0.5.*0.3).*(1-0.1.*(1-0.6.*0.5))).*0.8=0.15084

Matlab group exercise

- Test our result for this circuit.
- Download circuit_template.m from the website

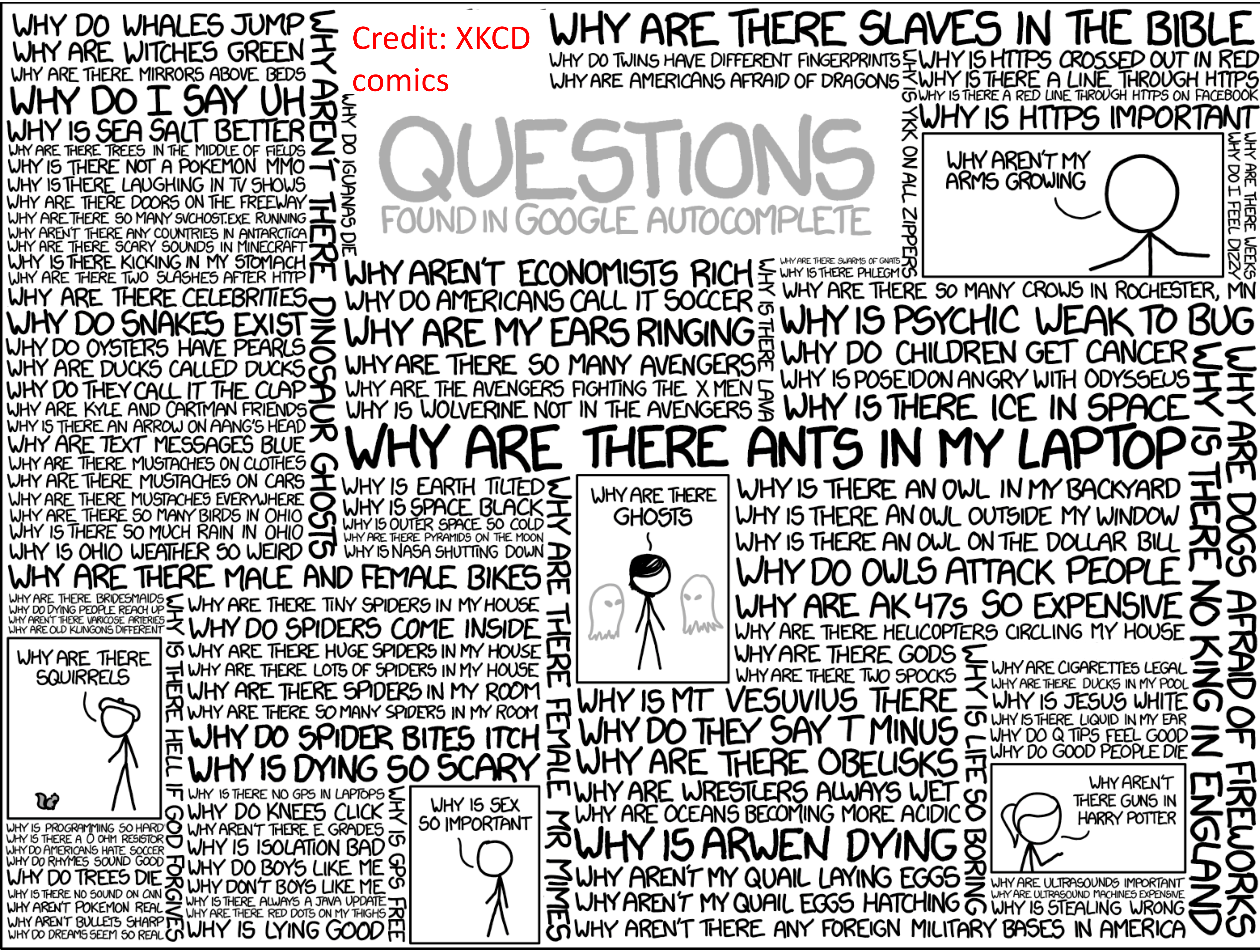


Component	E_1	E_2	E_3	E_4	E_5	E_6	E_7
Probability of functioning well	0.9	0.5	0.3	0.1	0.4	0.5	0.8

P(Works) = 0.9.*(1-(1-0.5.*0.3).*(1-0.1.*(1-0.6.*0.5))).*0.8=0.15084

Here is how I did it

- Stats=1e6;
- count= 0;
- for i = 1: Stats
- e1 = rand < 0.9; e2 = rand < 0.5; e3 = rand < 0.3;
- e4 = rand < 0.1; e5 = rand < 0.4; e6 = rand < 0.5;
- e7 = rand < 0.8;
- s1 = min(e2,e3); % or s1 = e2*e3;
- s2 = max(e5,e6); % or s2= e5+e6>0;
- s3 = min(e4,s2); % or s3 = e4*s2;
- s4 = max(s1,s3); % or s4 = s1+s3 > 0;
- s5= min([e1;s4;e7]); % or s5=e1*s4*e7;
- count = count + s5;
- End;
- P_circuit_works = count/Stats
- **% our calculation: P(circuit_works)= 0.9.*(1-(1-0.5.*0.3).*(1-0.1.*(1-0.6.*0.5))).*0.8==0.15084**

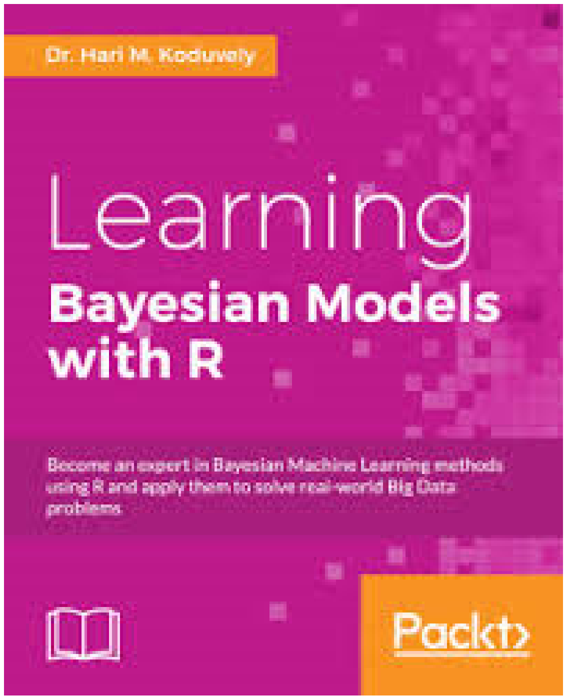


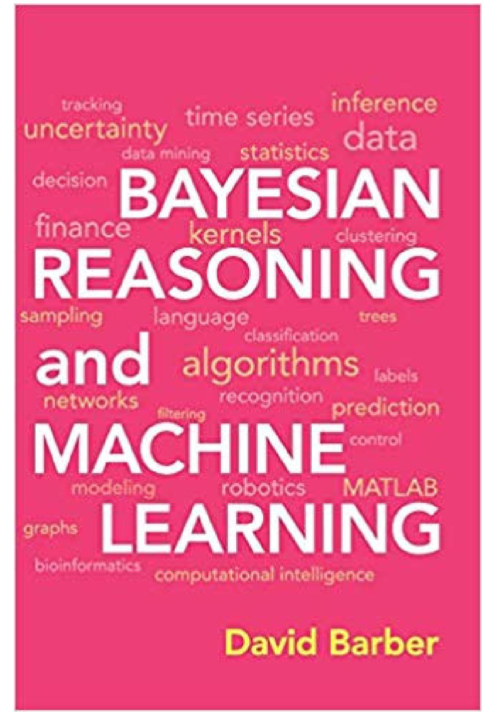
Credit: XKCD comics

Reminder:
Conditional probability

Bayes Theorem

Bayes’ theorem





Thomas Bayes (1701-1761)

English statistician, philosopher, and Presbyterian minister

Bayes’ theorem was presented in "An Essay towards solving a Problem in the Doctrine of Chances" which was read to the Royal Society in 1763 already after Bayes' death.

Bayes' theorem (simple)

$$P(A \cap B) = \underline{P(A|B)P(B)} = P(B \cap A) = \underline{P(B|A)P(A)}$$

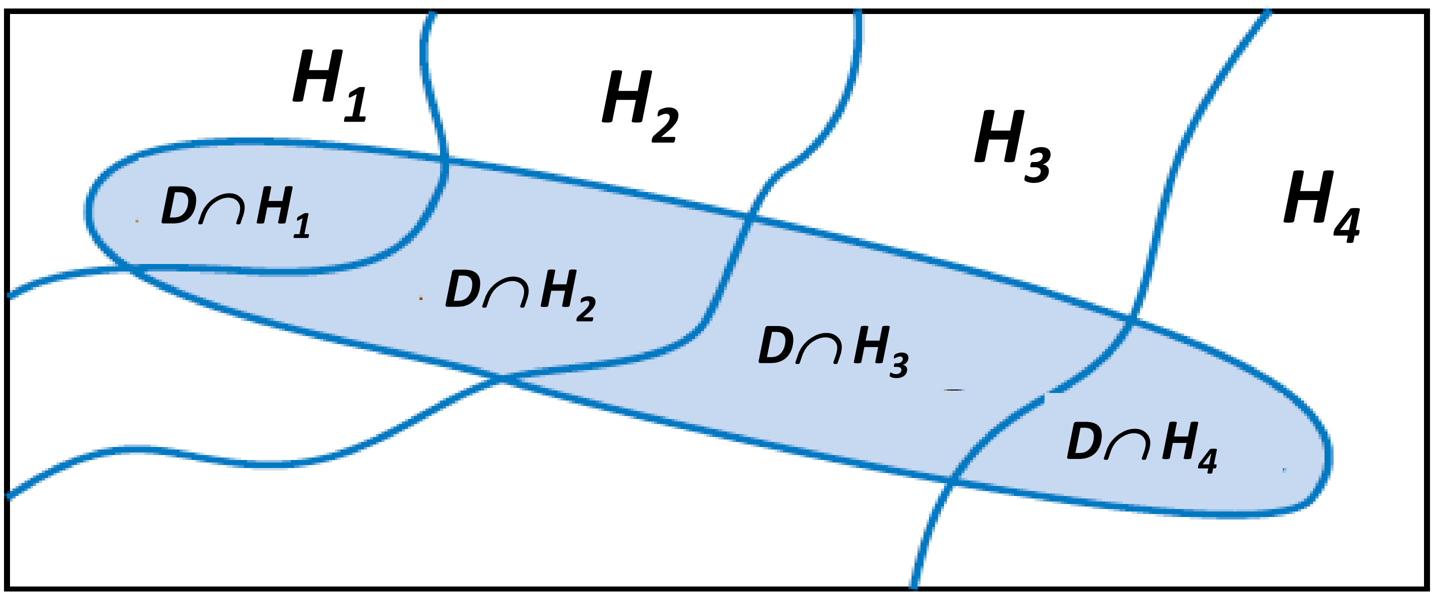
$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

- In Science we often want to know:
  "How much faith should I put into hypothesis, given the data?"
  or $P(H|D)$ (see also the inductive definition of probability)
- What we usually can calculate if the hypothesis/model is OK:
  "Assuming that this hypothesis is true, what is the probability of the observed data?" or $P(D|H)$
- Bayes' theorem can help: $P(H|D)=P(D|H)\bullet P(H)/P(D)$
- The problem is $P(H)$ (so-called prior) is often not known

Bayes' theorem (continued)

Works best with exhaustive and mutually-exclusive hypotheses:
$H_1, H_2, \ldots H_n$ such that $H_1 \cup H_2 \cup H_3 \ldots \cup H_n = S$ and $H_i \cap H_j = \varnothing$ for $i \neq j$

$P(H_k|D)=P(D|H_k) \bullet P(H_k)/P(D)$
where:
$P(D)= P(D|H_1) \bullet P(H_1) + P(D|H_2) \bullet P(H_2) + \ldots P(D|H_n) \bullet P(H_n)$



An **awesome new test** has been invented for an early detection of cancer. The probability that it correctly identifies someone with cancer as positive is 95%, and the probability that it correctly identifies someone without cancer as negative is 99%. The incidence of this type of cancer in the general population is 10^{-4}. A random person in the population takes the test, and the result is positive.

What is the probability that he/she has cancer?

A. 99%
B. 95%
C. 30%
D. 1%

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What is the probability that he/she has cancer?

A. 99%
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D. 1%

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participants

10^6

100 cancer — 95 positive tests

$10^6 - 100 \approx 10^6$ no cancer

10^6 participants with no cancer $\to$ 10,000 positive tests

$$P(C|P) = \frac{95}{10{,}000 + 95} \approx 1\%$$

Events: C - cancer, C' - no cancer

Test events P - positive, N - negative

We know:

$P(C) = 10^{-4}$, $\quad P(P|C) = 0.95$

$P(N|C') = 0.99$

We need

$P(C|P)$

Bayes:

$$P(C|P) = P(P|C) \cdot \frac{P(C)}{P(P)} \quad ?$$

$P(P)$ – probability that a random person will test positive

$$P(P) = P(P \cap C) + P(P \cap C') =$$

$$= P(P|C)P(C) + P(P|C')P(C') =$$

$$= 0.95 \times 10^{-4} + (1 - 0.99) \times (1 - 10^{-4}) \approx$$

$$\approx 10^{-4} + 10^{-2} \approx 10^{-2} = 1\%$$

$$P(C|P) = P(P|C) \cdot \frac{P(C)}{P(P)} = 0.95 \times \frac{10^{-4}}{10^{-2}} \approx$$

$$\approx 1\%$$

An **awesome new test** has been invented for an early detection of cancer. The probability that it correctly identifies someone with cancer as positive is 95%, and the probability that it correctly identifies someone without cancer as negative is 99%. The incidence of this type of cancer in the general population is 10^{-4}. A suspected cancer patient with likelihood of cancer 50% takes the test, and the result is positive.

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B. 95%
C. 30%
D. 1%

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An **awesome new test** has been invented for an early detection of cancer. The probability that it correctly identifies someone with cancer as positive is 95%, and the probability that it correctly identifies someone without cancer as negative is 99%. The incidence of this type of cancer in the general population is 10^{-4}. A suspected cancer patient with likelihood of cancer 50% takes the test, and the result is positive.

What is the probability that he/she has cancer?

A. 99%
B. 95%
C. 30%
D. 1%

Get your i-clickers

What if a doctor is already 50% sure of cancer based on other tests?

That changes things!

Now $P(C) = P(C') = 0.5$

$$P(C \mid P) = \frac{P(P \mid C) \cdot P(C)}{P(P \mid C) \cdot P(C) + P(P \mid C')\, P(C')} =$$

$$= \frac{0.95 \times 0.5}{0.95 \times 0.5 + (1 - 0.99) \times 0.5} \approx 0.99$$

How come?

I thought it was a ***great*** test..

- Let C – be the event that the patient has cancer; C' – patient is cancer free
- P/N – events that test is Positive/Negative (N=Y')
- We know: $P(C)=10^{-4}$, $P(P|C)=0.95$, $P(N|C')=0.99$
- We need to find $P(C|P)$
- Bayes to the rescue: $P(C|P)=P(P|C)*P(C)/P(P)$
- What on earth is P(P) ???

What on Earth is P(P) ???

- Likelihood that a random patient would test Y: $P(P)=P(P \cap C)+P(P \cap C')=P(P|C)P(C)+P(P|C')P(C')=$ $0.95*10^{-4}+(1-0.99)*(1-10^{-4})\approx 0.01$
- Hence $P(C|P)=P(P|C)*P(C)/P(P)$ $\approx 10^{-4}/0.01=0.01=1\%$
- But we would like it to be 100%, please!!!
- Until the false positive discovery rate $1-P(N|C')$ does not fall below the general population prevalence the <u>result will never be close 100%</u>

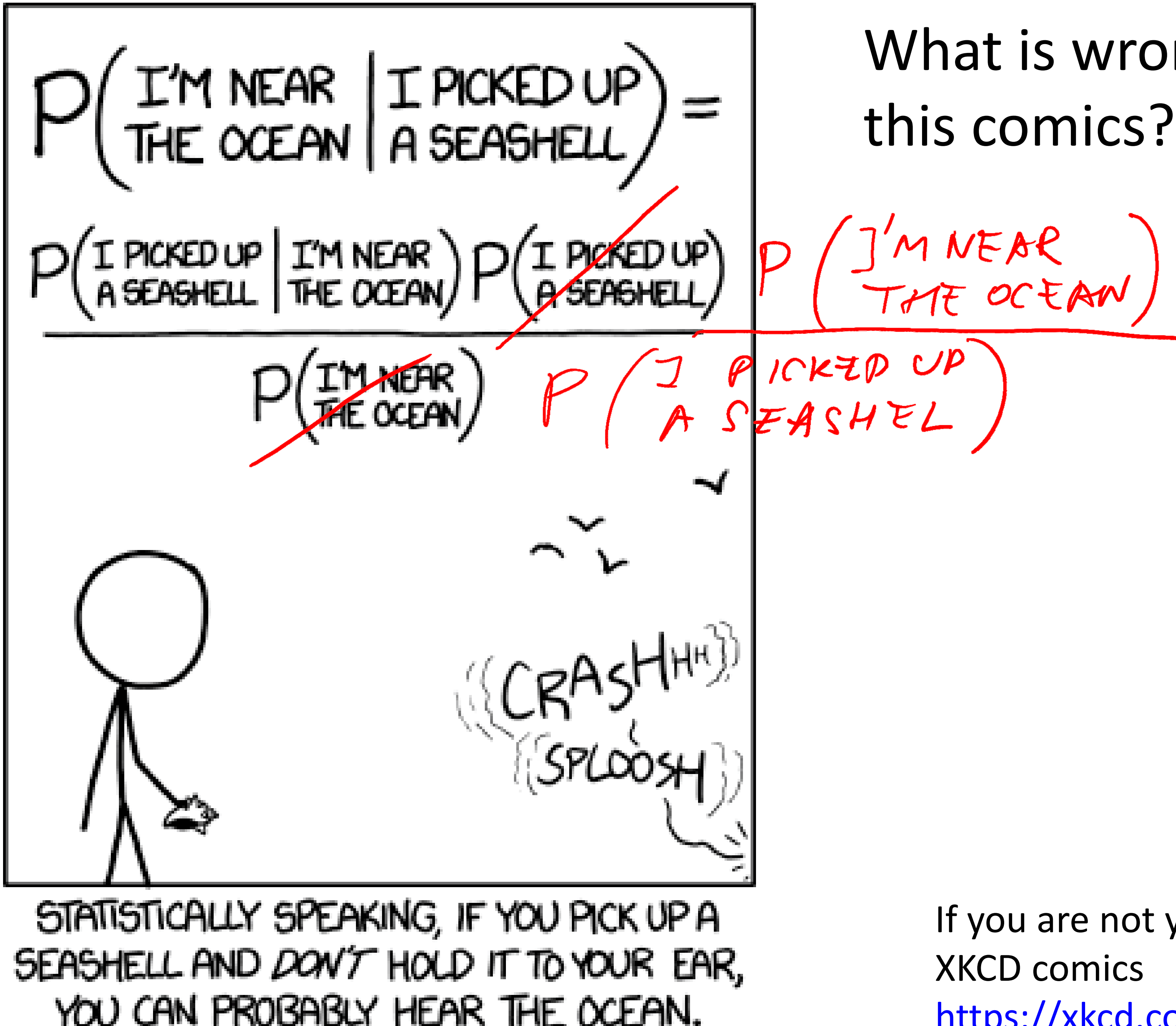
What if I am already 50% sure (based on other tests) that a patient has cancer?

- That changes everything!
- Now P(C)=P(C')=0.5
- P(C|P)=P(P|C)*P(C)/[P(P|C)*P(C)+ P(P|C')*P(C')]=

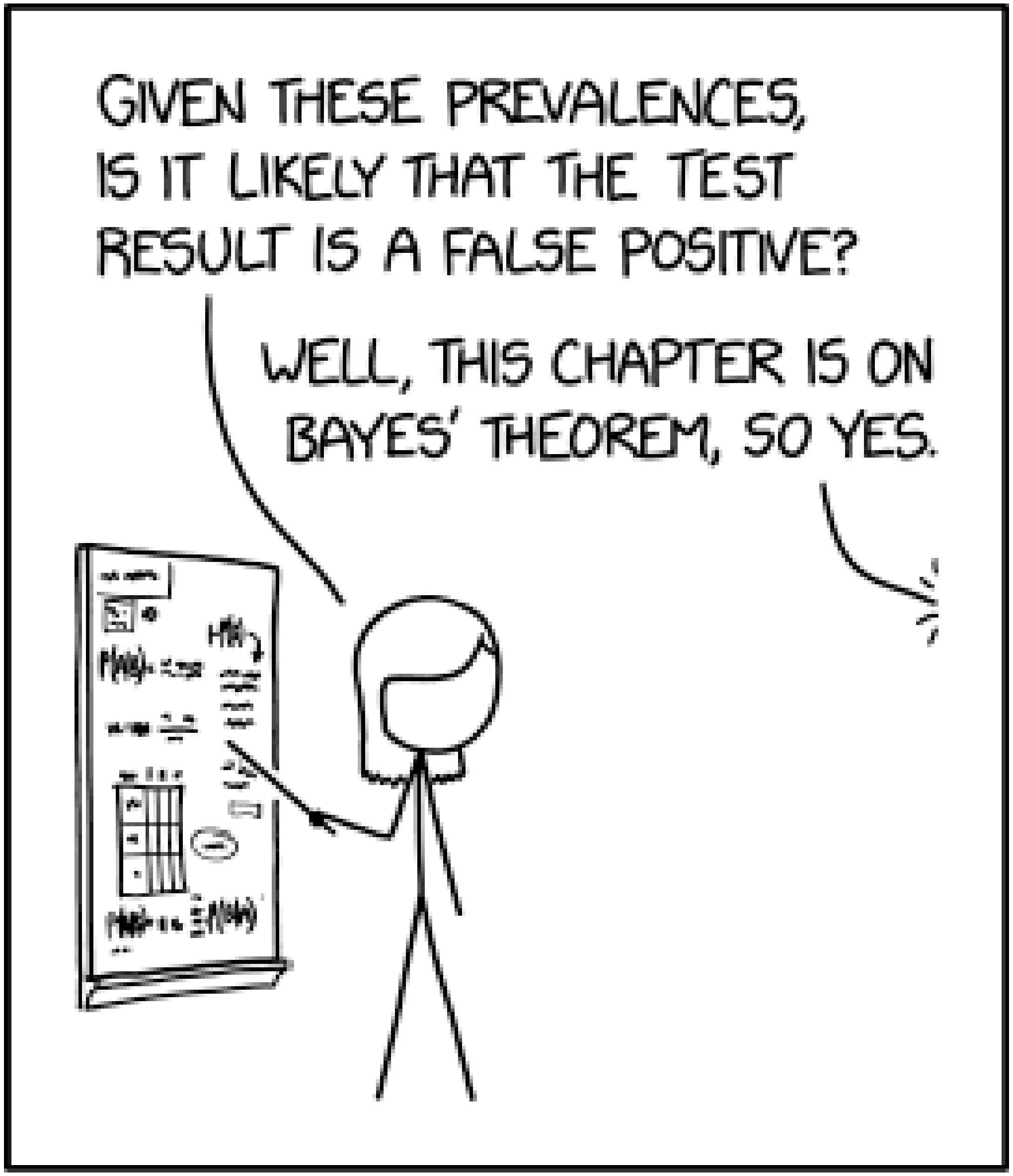
0.95*0.5/[0.95*0.5+(1-0.99)*0.5]=0.99

- Now the doctor can be almost 100% sure.
- The importance of prior:
  - If prior belief that one has cancer is 10^{-4} – test is useless
  - If prior belief is at least 1% - the test is useful

What is wrong in this comics?



If you are not yet reading XKCD comics https://xkcd.com/ you should start



SOMETIMES, IF YOU UNDERSTAND BAYES' THEOREM WELL ENOUGH, YOU DON'T NEED IT.

If you are not yet reading XKCD comics https://xkcd.com/ you should start

Sensitivity/specificity of the standard test for prostate cancer: PSA level > 4.0ng/mL

- Sensitivity of the test is 71.9%
  - fraction of cancer patients who will test positive
  - False negative rate is 28.1%
- Specificity of the test is 90%
  - fraction of healthy patients who will test negative
  - False positive rate is 10%

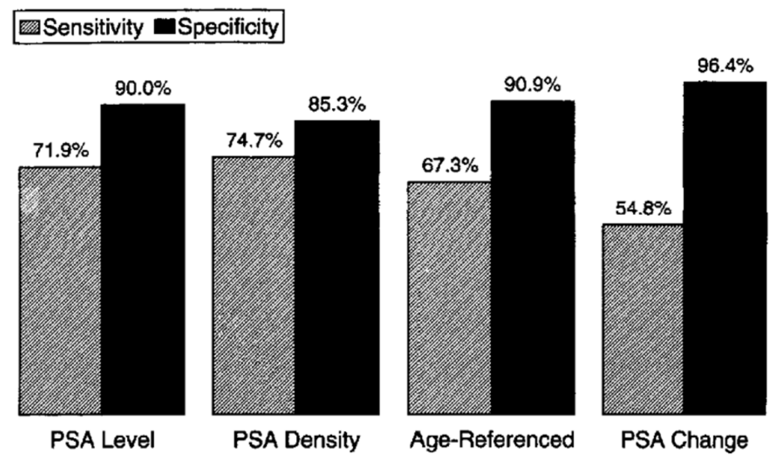


Figure 1. The relative sensitivity and specificity of different indexes of prostate specific antigen (PSA). Except for PSA change, sensitivity is the proportion of 171 known cancers cases for whom the index was positive and specificity is the proportion of 2011 men with normal transrectal ultrasound and digital rectal examinations not known to have prostate cancer who were negative on the index. The sensitivity and specificity of PSA change was evaluated in 84 men with prostate cancer and 1473 men without prostate cancer for on whom multiple PSA measures were available. A PSA level of 4.0 ng/ml or less was considered normal. A PSA density of 0.1 ng/ml per cubic centimeter of ultrasound-measured gland volume was considered normal. Age-referenced PSA was considered normal if it was 3.5 ng/ml or less in men aged 50–59, 4.5 ng/ml in men aged 60–69, and 6.5 ng/ml in men aged 70–79. PSA change was considered normal if the annual rate of PSA change was [illegible] or less per year.

Mettlin C, Littrup PJ, Kane RA, Murphy GP, Lee F, Chesley A, et al. Cancer. 1994;74: 1615–1620.

Prostate cancer is the most common type of cancer found in males. It is checked by PSA test that is notoriously unreliable. The probability that a noncancerous man will have an elevated PSA level >4.0 ng/mL is approximately 0.1, with this probability increasing to approximately 0.719 if the man does have prostate cancer. If, based on other factors, a physician is 50 percent certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that the test indicates an elevated PSA level?

A. 99.9%
B. 95%
C. 88%
D. 55%

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All this trouble for a lousy 38% gain in confidence? I don't believe you!

- Let C – be the event that the patient has cancer; C' – patient is cancer free, E – events that the PSA test was elevated
- We know **doctor's prior belief**: $P(C)=0.5$
- We know test stats: $P(E|C)=0.719$, $P(E|C')=0.1$
- We need to find $P(C|E)=P(E|C)*P(C)/P(E)$
- $P(E)=P(E|C)*P(C)+P(E|C')*P(C')=$
  $=0.719*0.5+0.1*0.5=0.41$
- $P(C|E)=0.5*0.719/0.41=0.88$ or 88%

Credit: XKCD comics