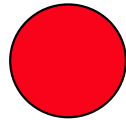


Inductive probability
relies on combinatorics
or the art of counting
combinations

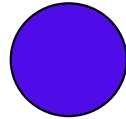
Multiplication and permutation
rules are two examples
of a general
problem, where
a sample of size k is drawn
from a population of
 n distinct objects

Balls drawn from an urn (or a box)

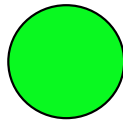
1 ball is red



1 ball is blue



1 ball is green

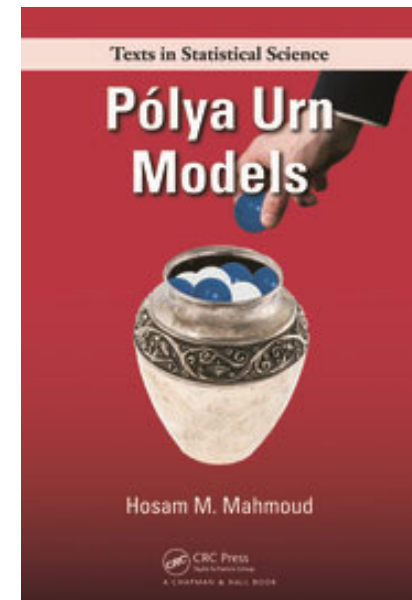


$n=3$ balls of different colors in a box from which I draw $k=2$ balls one at a time

- Do I put each ball back to the box after drawing it?
 - Yes: problem with replacement
 - No: problem without replacement
- Do I keep track of the order in which the balls were drawn?
 - Yes: the order matters
 - No: the order does not matter

George Pólya

- George Pólya (December 13, 1887 – September 7, 1985) was a Hungarian mathematician.
- He was a professor of mathematics at ETH Zürich from 1914 to 1940 and at Stanford University from 1940 to 1953. He made fundamental contributions to combinatorics, number theory, numerical analysis and probability theory.

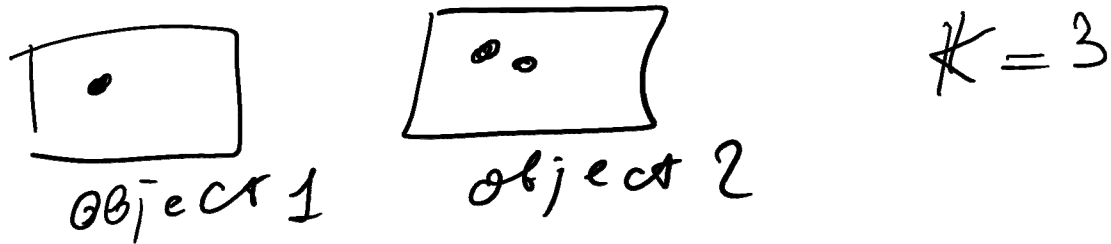


How many ways to choose a sample of k objects out of a population of n objects

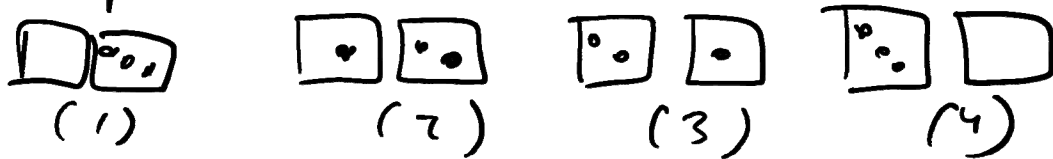
	Order matters	Order does not matter
replace	$n \times n \times n \times \dots \times n$ $= n^k$	$\frac{n^k}{k!}$ <p>not all objects are different</p>
Do not replace	$n \times (n-1) \times$ $\times (n-2) \times \dots \times$ $(n-k+1) =$ $= \frac{n!}{(n-k)!}$	<p>All objects are different \rightarrow</p> $\frac{n!}{(n-k)!} \times \frac{1}{k!} = \binom{n}{k}$

How to solve the problem of K out of n with replacement but where order does not matter?

Let's solve $n=2$ problem first:



4 possibilities



$\bullet\bullet || \bullet\bullet\bullet | \bullet\bullet$ K dots, $n-1$ box boundaries

$$\binom{k+n-1}{k} = \frac{(k+n-1)!}{k! (n-1)!} \quad \text{ways to distribute}$$

Sampling table

How many ways to choose a **sample of k objects** out of **population of n objects**?

	Order matters	Order does not matter
Replacement	$(n)^k$	Difficult: $\binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!k!}$
No replacement	$n(n-1)(n-2)..(n-k+1) = \frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Example

- A DNA of 100 bases is characterized by its numbers of 4 nucleotides:
 $d_A, d_C, d_G,$ and d_T ($d_A + d_C + d_G + d_T = 100$)
- **I don't care about the sequence** (only about the total numbers of A,C,G, and T)
- How many distinct combinations of $d_A, d_C, d_G,$ and d_T are out there?

What is n and k in this problem?

- A. $k=100, n=4$
- B. $k=4, n=100$
- C. $k=100, n=4^{100}$
- D. $k=4^{100}, n=4$
- E. I don't know

Get your i-clickers

What is n and k in this problem?

A. $k=100$, $n=4$

B. $k=4$, $n=100$

C. $k=100$, $n=4^{100}$

D. $k=4^{100}$, $n=4$

E. I don't know

Is it sampling **with or without replacement** & **does order matter?**

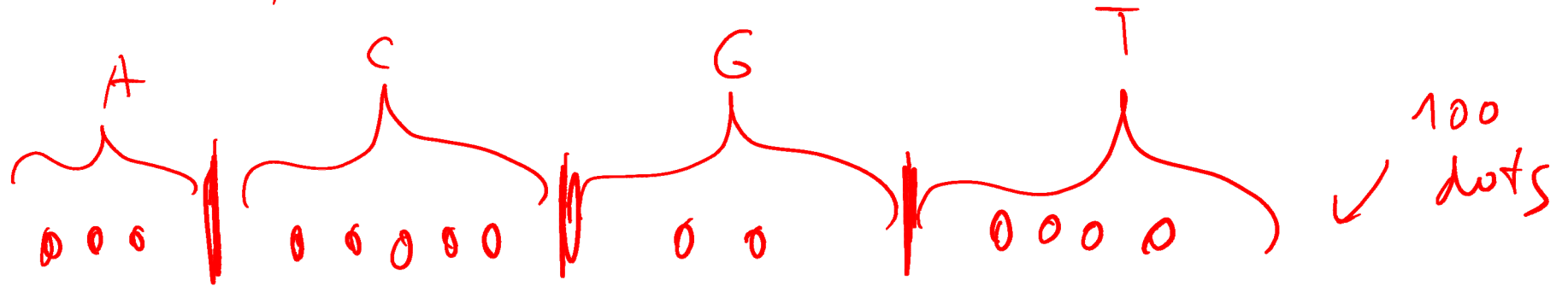
- A. **with replacement, order matters**
- B. **without replacement, any order**
- C. **with replacement, any order**
- D. **without replacement, order matters**
- E. I don't know

Get your i-clickers

Is it sampling **with or without replacement** & **does order matter?**

- A. **with replacement, order matters**
- B. **without replacement, any order**
- C. **with replacement, any order**
- D. **without replacement, order matters**
- E. I don't know

$$n = 4, k = 100$$



$$\# \text{ of possibilities} = \frac{(100 + 4 - 1)!}{(4 - 1)! \cdot 100!} =$$

$$= \frac{103 \cdot 102 \cdot 101}{3 \cdot 2 \cdot 1} = 176,851$$

If order did matter
 $4^{100} = 2^{200} = 10^{60} \approx \# \text{ atoms in a galaxy}$

Probability Axioms,
Conditional Probability,
Statistical (In)dependence,
Circuit Problems

Axioms of probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

(3) For two events E_1 and E_2 with $E_1 \cap E_2 = \emptyset$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

$$P(\emptyset) = 0$$

These axioms imply that:

$$P(E') = 1 - P(E)$$

if the event E_1 is contained in the event E_2

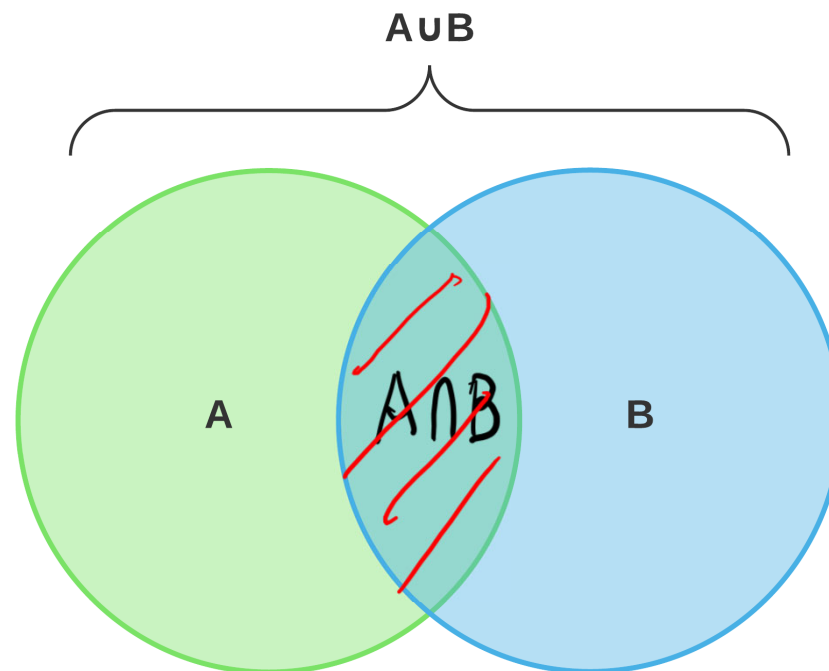
$$P(E_1) \leq P(E_2)$$

Addition rules following from the Axiom (3)

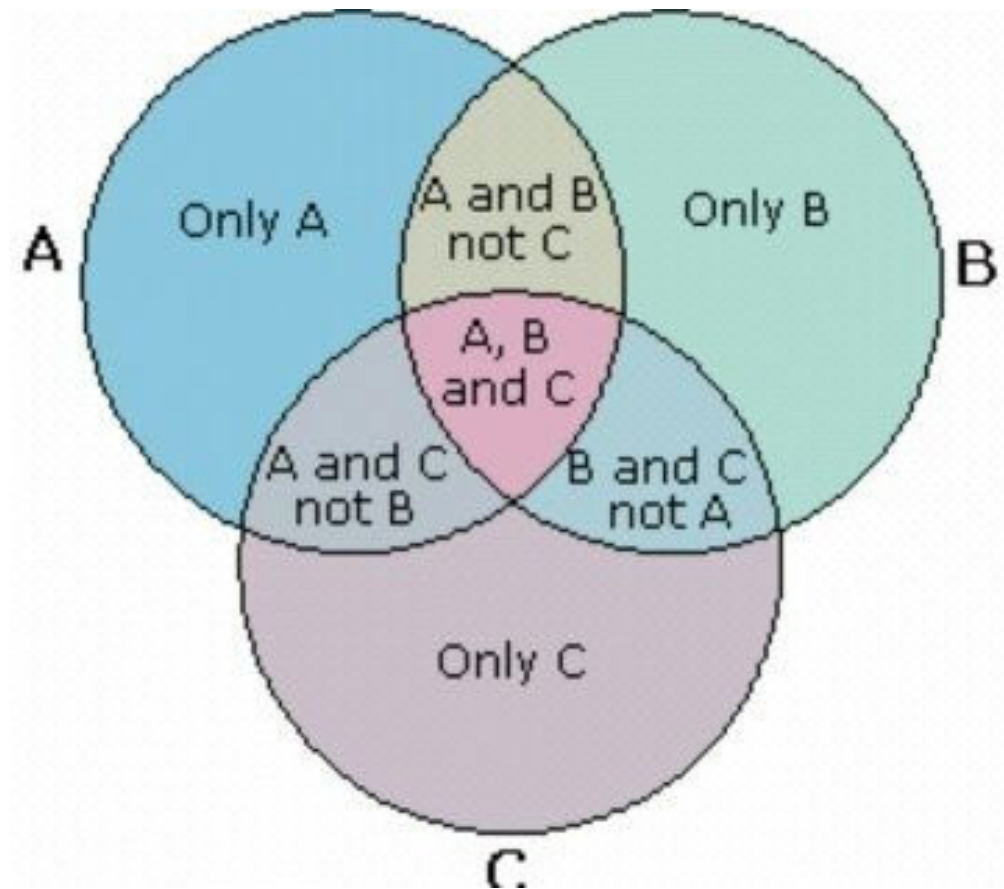
If A and B are mutually exclusive events, i.e. $A \cap B = \emptyset$

$$P(A \cup B) = P(A) + P(B) \quad (2-2)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (2-1)$$



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) -$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C) +$$
$$+ P(A \cap B \cap C).$$



Conditional probability

The **conditional probability** of an event B given an event A , denoted as $P(B|A)$, is

$$P(B|A) = P(A \cap B)/P(A)$$

for $P(A) > 0$.

This definition can be understood in a special case in which all outcomes of a random experiment are equally likely. If there are n total outcomes,

$$P(A) = (\text{number of outcomes in } A)/n$$

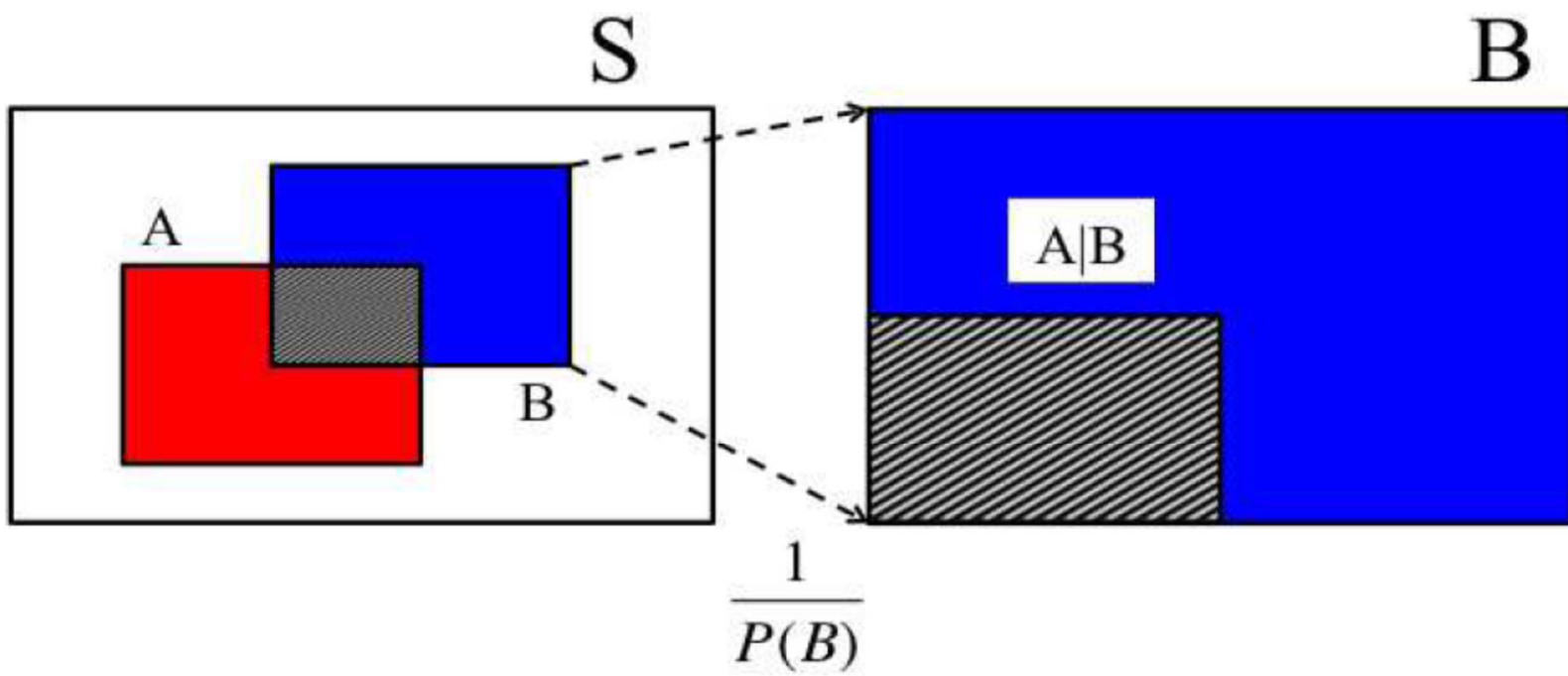
Also,

$$P(A \cap B) = (\text{number of outcomes in } A \cap B)/n$$

Consequently,

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

Therefore, $P(B|A)$ can be interpreted as the relative frequency of event B among the trials that produce an outcome in event A .



Multiplication rule

is just definition of conditional probability

$$P(\mathbf{B} \mid \mathbf{A}) = P(\mathbf{B} \cap \mathbf{A}) / P(\mathbf{A}) \rightarrow$$

$$P(\mathbf{B} \cap \mathbf{A}) = P(\mathbf{B} \mid \mathbf{A}) \cdot P(\mathbf{A})$$

Drake equation

$$N = R^* \cdot f_p \cdot n_e \cdot f_l \cdot f_i \cdot f_c \cdot L$$

- N = The number of civilizations in The Milky Way Galaxy whose electromagnetic emissions are detectable.
- R^* = The rate of formation of stars suitable for the development of intelligent life.
- f_p = The fraction of those stars with planetary systems.
- n_e = The number of planets, per solar system, with an environment suitable for life.
- f_l = The fraction of suitable planets on which life actually appears.
- f_i = The fraction of life bearing planets on which intelligent life emerges.
- f_c = The fraction of civilizations that develop a technology that releases detectable signs of their existence into space.
- L = The length of time such civilizations release them

Ms. Perez figures that there is a 5% chance that her company will set up a branch in Phoenix. If it does, she is 10% certain that she will be made its manager. What is the probability that Perez will be a Phoenix branch office manager?

A. 15%

B. 0.5%

C. 50%

D. 5%

E. 10%

Get your i-clickers

Statistically independent events

Always true: $P(A \cap B) = P(A | B) \cdot P(B) = P(B | A) \cdot P(A)$

■ Two events

Two events are **independent** if **any one** of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

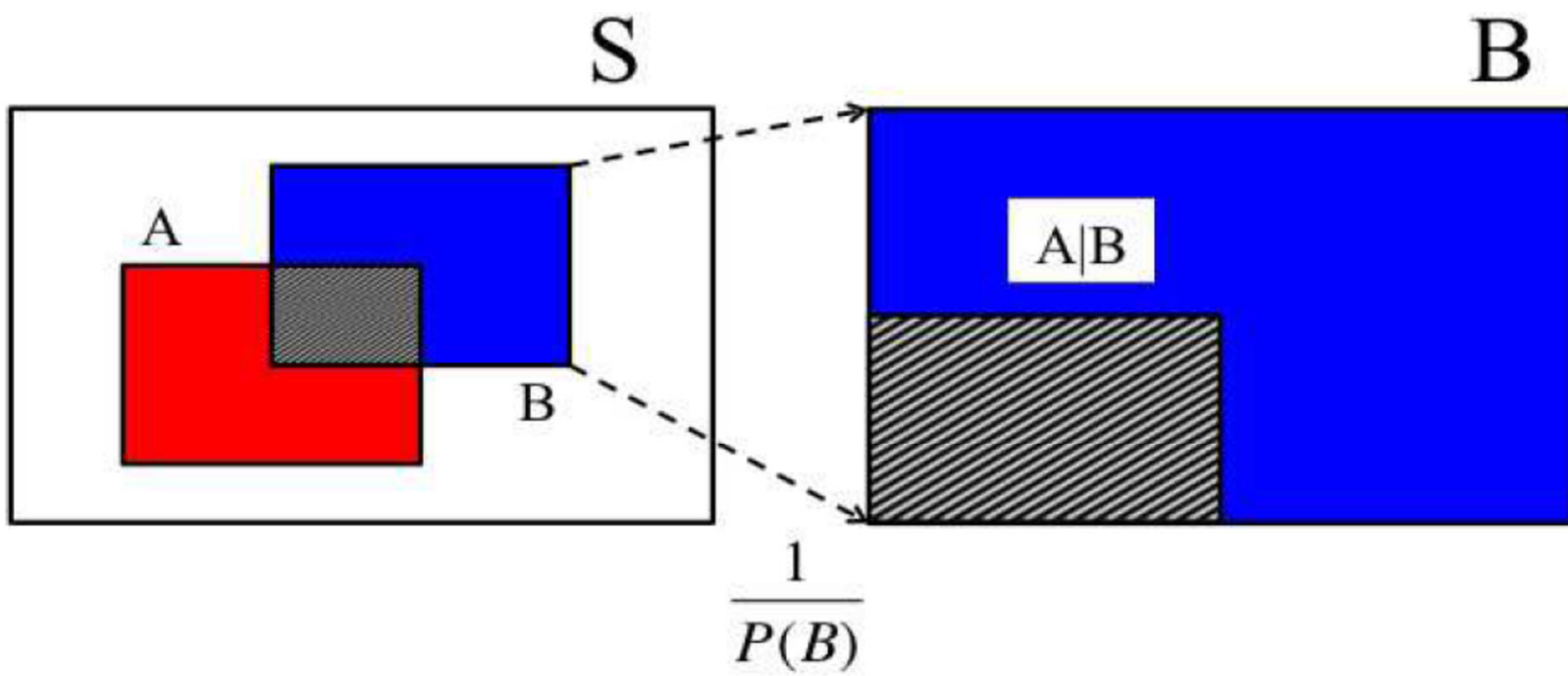
(2) $P(B|A) = P(B)$

(3) $P(A \cap B) = P(A)P(B)$

■ Multiple events

The events E_1, E_2, \dots, E_n are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \dots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \dots \times P(E_{i_k})$$



Example 3.10. Let an experiment consist of drawing a card at random from a standard deck of 52 playing cards. Define events A and B as “the card is a ♠” and “the card is a queen.” Are the events A and B independent? By definition, $P(A \cdot B) = P(Q \spadesuit) = \frac{1}{52}$. This is the product of $P(\spadesuit) = \frac{13}{52}$ and $P(Q) = \frac{4}{52}$, and events A and B in question are independent. In this situation, intuition provides no help. Now, pretend that the $2\heartsuit$ is drawn and excluded from the deck prior to the experiment. Events A and B become dependent since

$$\mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{13}{51} \cdot \frac{4}{51} \neq \frac{1}{51} = \mathbb{P}(A \cdot B).$$

Credit: XKCD
comics

WHY ARE THERE SLAVES IN THE BIBLE

WHY DO TWINS HAVE DIFFERENT FINGERPRINTS
WHY ARE AMERICANS AFRAID OF DRAGONS

WHY IS HTTPS CROSSED OUT IN RED
WHY IS THERE A LINE THROUGH HTTPS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT

QUESTIONS

FOUND IN GOOGLE AUTOCOMPLETE



WHY ARE THERE WEEKS
WHY DO I FEEL DIZZY

WHY AREN'T ECONOMISTS RICH

WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN
WHY IS THERE PHLEGM

WHY DO AMERICANS CALL IT SOCCER

WHY IS PSYCHIC WEAK TO BUG

WHY ARE MY EARS RINGING

WHY DO CHILDREN GET CANCER

WHY ARE THERE SO MANY AVENGERS

WHY IS POSEIDON ANGRY WITH ODYSSEUS

WHY ARE THE AVENGERS FIGHTING THE X MEN

WHY IS THERE ICE IN SPACE

WHY ARE THERE ANTS IN MY LAPTOP

WHY IS EARTH TILTED

WHY ARE THERE GHOSTS

WHY IS THERE AN OWL IN MY BACKYARD

WHY IS SPACE BLACK

WHY ARE THERE GHOSTS

WHY IS THERE AN OWL OUTSIDE MY WINDOW

WHY IS OUTER SPACE SO COLD

WHY ARE THERE GHOSTS

WHY IS THERE AN OWL ON THE DOLLAR BILL

WHY ARE THERE PYRAMIDS ON THE MOON

WHY ARE THERE GHOSTS

WHY DO OWLS ATTACK PEOPLE

WHY IS NASA SHUTTING DOWN

WHY ARE THERE GHOSTS

WHY ARE AK 47s SO EXPENSIVE

WHY ARE THERE MALE AND FEMALE BIKES

WHY ARE THERE GHOSTS

WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE

WHY ARE THERE TINY SPIDERS IN MY HOUSE

WHY ARE THERE GHOSTS

WHY ARE THERE GODS

WHY DO SPIDERS COME INSIDE

WHY ARE THERE GHOSTS

WHY ARE THERE TWO SPOCKS

WHY ARE THERE HUGE SPIDERS IN MY HOUSE

WHY ARE THERE GHOSTS

WHY IS LIFE SO BORING

WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE

WHY ARE THERE GHOSTS

WHY ARE CIGARETTES LEGAL

WHY ARE THERE SPIDERS IN MY ROOM

WHY ARE THERE GHOSTS

WHY ARE THERE DUCKS IN MY POOL

WHY ARE THERE SO MANY SPIDERS IN MY ROOM

WHY ARE THERE GHOSTS

WHY IS JESUS WHITE

WHY DO SPIDER BITES ITCH

WHY ARE THERE GHOSTS

WHY IS THERE LIQUID IN MY EAR

WHY IS DYING SO SCARY

WHY ARE THERE GHOSTS

WHY DO Q TIPS FEEL GOOD

WHY DO WHALES JUMP
WHY ARE WITCHES GREEN
WHY ARE THERE MIRRORS ABOVE BEDS

WHY AREN'T THERE DINOSAUR GHOSTS

WHY DO I SAY UH
WHY IS SEA SALT BETTER

WHY ARE THERE TREES IN THE MIDDLE OF FIELDS
WHY IS THERE NOT A POKEMON MMO
WHY IS THERE LAUGHING IN TV SHOWS

WHY ARE THERE DOORS ON THE FREEWAY
WHY ARE THERE SO MANY SVCHOST.EXE RUNNING
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA

WHY ARE THERE SCARY SOUNDS IN MINECRAFT
WHY IS THERE KICKING IN MY STOMACH
WHY ARE THERE TWO SLASHES AFTER HTTP

WHY ARE THERE CELEBRITIES
WHY DO SNAKES EXIST
WHY DO OYSTERS HAVE PEARLS

WHY ARE DUCKS CALLED DUCKS
WHY DO THEY CALL IT THE CLAP
WHY ARE KYLE AND CARTMAN FRIENDS
WHY IS THERE AN ARROW ON AANG'S HEAD

WHY ARE TEXT MESSAGES BLUE
WHY ARE THERE MUSTACHES ON CLOTHES
WHY ARE THERE MUSTACHES ON CARS
WHY ARE THERE MUSTACHES EVERYWHERE

WHY ARE THERE SO MANY BIRDS IN OHIO
WHY IS THERE SO MUCH RAIN IN OHIO
WHY IS OHIO WEATHER SO WEIRD

WHY ARE THERE BRIDESMAIDS
WHY DO DYING PEOPLE REACH UP
WHY AREN'T THERE VARICOSE ARTERIES
WHY ARE OLD KUNGONS DIFFERENT

WHY ARE THERE SQUIRRELS

WHY IS PROGRAMMING SO HARD
WHY IS THERE A 0 OHM RESISTOR
WHY DO AMERICANS HATE SOCCER
WHY DO RHYMES SOUND GOOD

WHY DO TREES DIE
WHY IS THERE NO SOUND ON CNN
WHY AREN'T POKEMON REAL
WHY AREN'T BULLETS SHARP
WHY DO DREAMS SEEM SO REAL

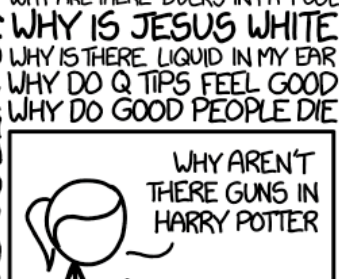
WHY DO IGUANAS DIE

WHY AREN'T THERE DINOSAUR GHOSTS

WHY ARE THERE FEMALE MR NIMES

WHY IS GPS FREE

WHY IS GPS FREE

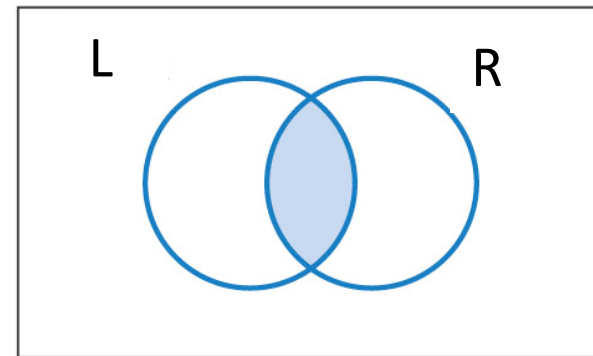


WHY ARE DOGS AFRAID OF FIREWORKS
WHY IS THERE NO KING IN ENGLAND

WHY ARE ULTRASOUNDS IMPORTANT
WHY ARE ULTRASOUND MACHINES EXPENSIVE
WHY IS STEALING WRONG
WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA

Series Circuit

This circuit operates only if there is **at least one path of functional devices** from left to right. The **probability** that **each device functions** is shown on the graph. Assume that the **devices fail independently**. What is the probability that the circuit operates?

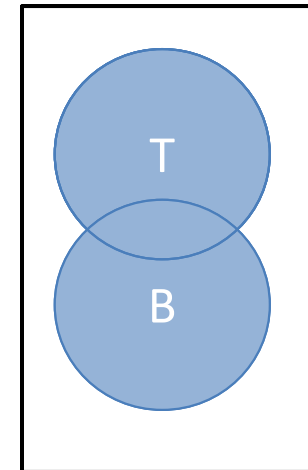
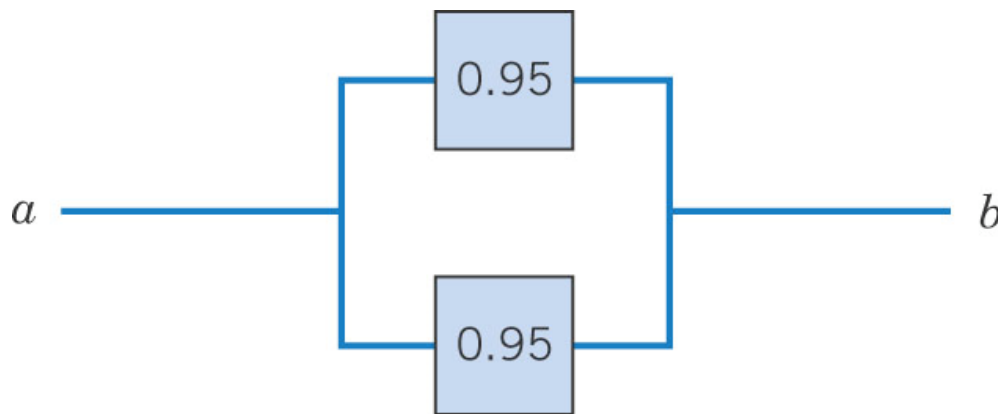


Let L & R denote the events that the left and right devices operate. The probability that the circuit operates is:

$$P(L \text{ and } R) = P(L \cap R) = P(L) * P(R) = 0.8 * 0.9 = 0.72.$$

Parallel Circuit

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown. Each device fails independently.

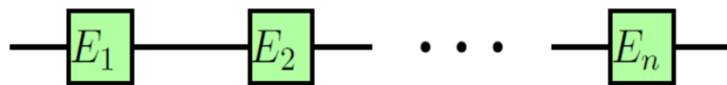


Let T & B denote the events that the top and bottom devices operate. The probability that the circuit operates is:

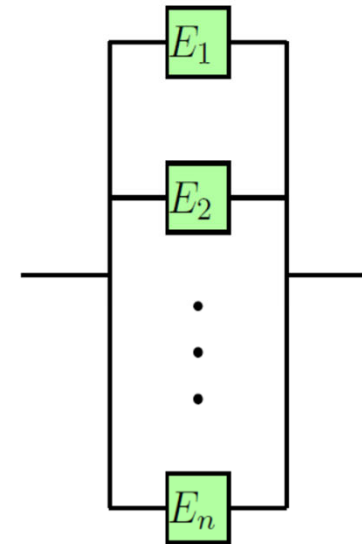
$$P(T \cup B) = 1 - P(T' \cap B') = 1 - P(T') * P(B') = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

Duality between parallel and series circuits

$$q_i = 1 - p_i.$$



(a)

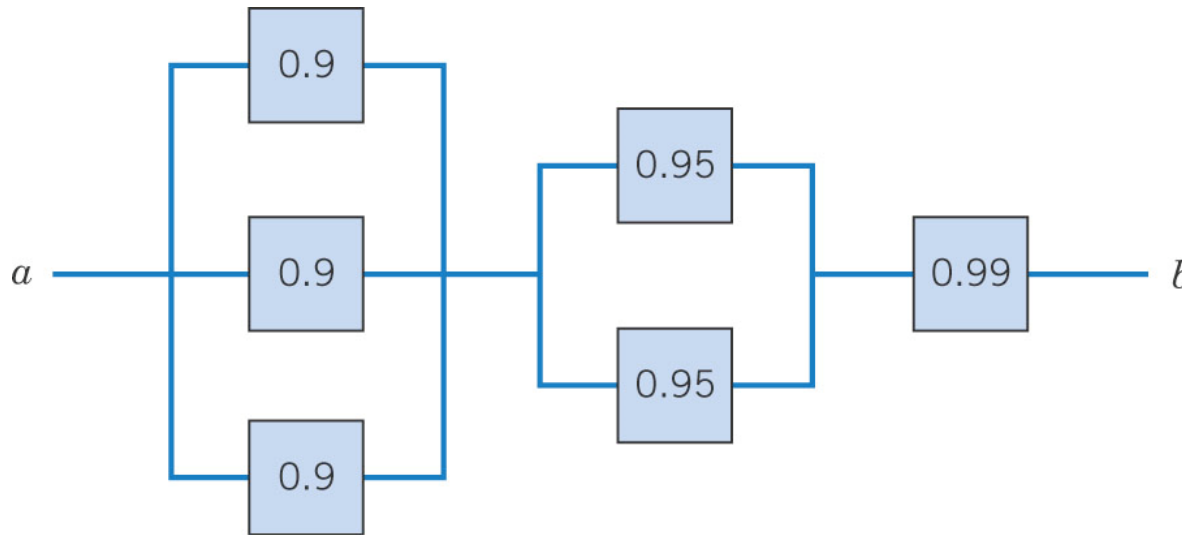


(b)

Connection	Notation	Works with prob	Fails with prob
Serial	$E_1 \cap E_2 \cap \dots \cap E_n$	$p_1 p_2 \dots p_n$	$1 - p_1 p_2 \dots p_n$
Parallel	$E_1 \cup E_2 \cup \dots \cup E_n$	$1 - q_1 q_2 \dots q_n$	$q_1 q_2 \dots q_n$

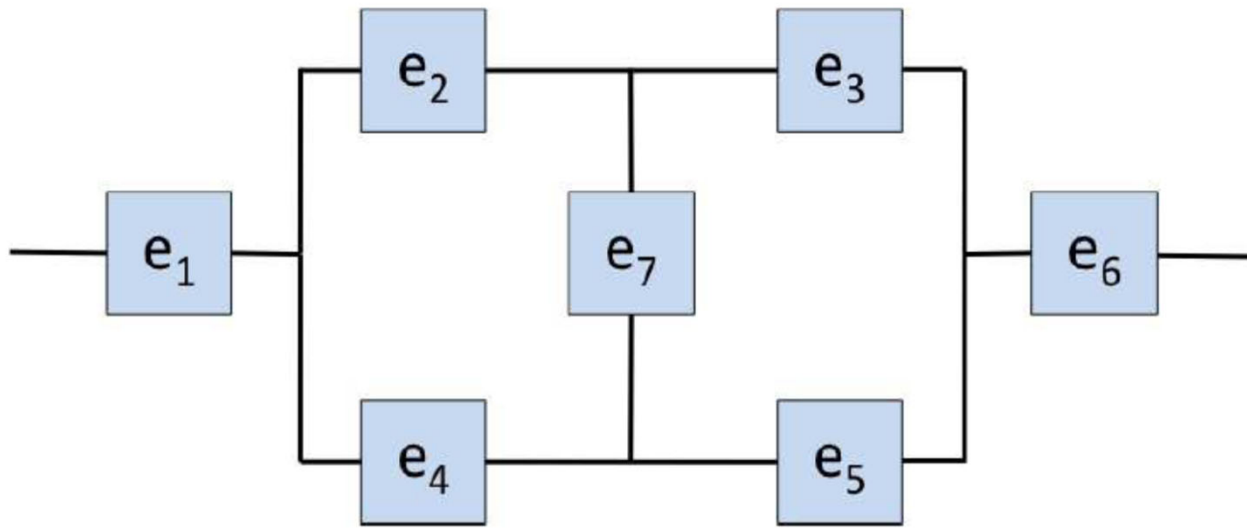
Advanced Circuit

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown. Each device fails independently.



Partition the graph into 3 columns with L & M denoting the left & middle columns.

$P(L) = 1 - 0.1^3$, and $P(M) = 1 - 0.05^2$, so the probability that the circuit operates is: $(1 - 0.1^3)(1 - 0.05^2)(0.99) = 0.9875$ (this is a series of parallel circuits).



Component	e_1	e_2	e_3	e_4	e_5	e_6	e_7
Probability of component working	0.3	0.8	0.2	0.2	0.5	0.6	0.4