

# Marginal Probability Distributions (discrete)

For a **discrete** joint PDF, there are **marginal distributions** for **each random variable**, formed by summing the joint PMF over the other variable.

$$f_X(x) = \sum_y f_{XY}(x, y)$$

$$f_Y(y) = \sum_x f_{XY}(x, y)$$

y = number of times city name is stated	x = number of bars of signal strength			$f_Y(y) =$
	1	2	3	
1	0.01	0.02	0.25	0.28
2	0.02	0.03	0.20	0.25
3	0.02	0.10	0.05	0.17
4	0.15	0.10	0.05	0.30
$f_X(x) =$	0.20	0.25	0.55	1.00

Called **marginal** because they are **written in the margins**

**Figure 5-6** From the prior example, the joint PMF is shown in green while the two marginal PMFs are shown in purple.

# Conditional Probability Distributions

Recall that  $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$$P(Y=y | X=x) = P(X=x, Y=y) / P(X=x) = f(x, y) / f_X(x)$$

From Example 5-1

$$P(Y=1 | X=3) = 0.25/0.55 = 0.455$$

$$P(Y=2 | X=3) = 0.20/0.55 = 0.364$$

$$P(Y=3 | X=3) = 0.05/0.55 = 0.091$$

$$P(Y=4 | X=3) = 0.05/0.55 = 0.091$$

$$\text{Sum} = 1.00$$

y = number of times city name is stated	x = number of bars of signal strength			$f_Y(y) =$
	1	2	3	
1	0.01	0.02	0.25	0.28
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4	0.15	0.10	0.05	0.30
$f_X(x) =$	0.20	0.25	0.55	1.00

Note that there are 12 probabilities conditional on  $X$ , and 12 more probabilities conditional upon  $Y$ .

X and Y are Bernoulli variables

	Y=0	Y=1
X=0	2/6	1/6
X=1	2/6	1/6

What is the marginal  $P_Y(Y=0)$ ?

A. 1/6

B. 2/6

C. 3/6

D. 4/6

E. I don't know

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# X and Y are Bernoulli variables

	Y=0	Y=1
X=0	2/6	1/6
X=1	2/6	1/6

What is the conditional  $P(X=0 | Y=1)$ ?

A.  $2/6$

B.  $1/2$

C.  $1/6$

D.  $4/6$

E. I don't know

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Reminder

# Statistically independent events

Always true:  $P(A \cap B) = P(A | B) \cdot P(B) = P(B | A) \cdot P(A)$

## ■ Two events

Two events are **independent** if **any one** of the following equivalent statements is true:

(1)  $P(A|B) = P(A)$

(2)  $P(B|A) = P(B)$

(3)  $P(A \cap B) = P(A)P(B)$

## ■ Multiple events

The events  $E_1, E_2, \dots, E_n$  are independent if and only if for any subset of these events  $E_{i_1}, E_{i_2}, \dots, E_{i_k}$ ,

$$P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \dots \times P(E_{i_k})$$

# Independence of Random Variables $X$ and $Y$

- **Random variable independence** means that knowledge of **any** of the values of  $X$  **does not change** probabilities of **any** of the values of  $Y$
- Opposite: **Dependence** implies that **some** values of  $X$  influence the probability of **some** values of  $Y$

# Independence for Discrete Random Variables

- Remember independence of events (slide 13 lecture 4) : Events are independent if **any one** of the three conditions are met:
  - 1)  $P(A|B) = P(A \cap B) / P(B) = P(A)$  or
  - 2)  $P(B|A) = P(A \cap B) / P(A) = P(B)$  or
  - 3)  $P(A \cap B) = P(A) \cdot P(B)$
- Random variables independent if **all events**  $A$  that  $Y=y$  and  $B$  that  $X=x$  are independent if any one of these conditions is met:
  - 1)  $P(Y=y | X=x) = P(Y=y)$  for any  $x$  or
  - 2)  $P(X=x | Y=y) = P(X=x)$  for any  $y$  or
  - 3)  $P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$**for every pair  $x$  and  $y$**



# X and Y are Bernoulli variables

	Y=0	Y=1
X=0	2/6	1/6
X=1	2/6	1/6

Are they independent?

A. yes

B. no

C. I don't know

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# X and Y are Bernoulli variables

	Y=0	Y=1
X=0	1/2	0
X=1	0	1/2

Are they independent?

A. yes

B. no

C. I don't know

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Credit: XKCD  
comics

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WHY IS DYING SO SCARY

WHY IS THERE NO GPS IN LAPTOPS  
WHY DO KNEES CLICK  
WHY AREN'T THERE E GRADES

WHY IS ISOLATION BAD  
WHY DO BOYS LIKE ME  
WHY DON'T BOYS LIKE ME

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WHY ARE THERE RED DOTS ON MY THIGHS  
WHY IS LYING GOOD

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WHY IS MT VESUVIUS THERE  
WHY DO THEY SAY T MINUS

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WHY ARE WRESTLERS ALWAYS WET

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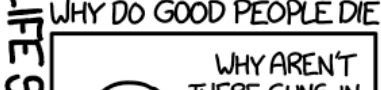
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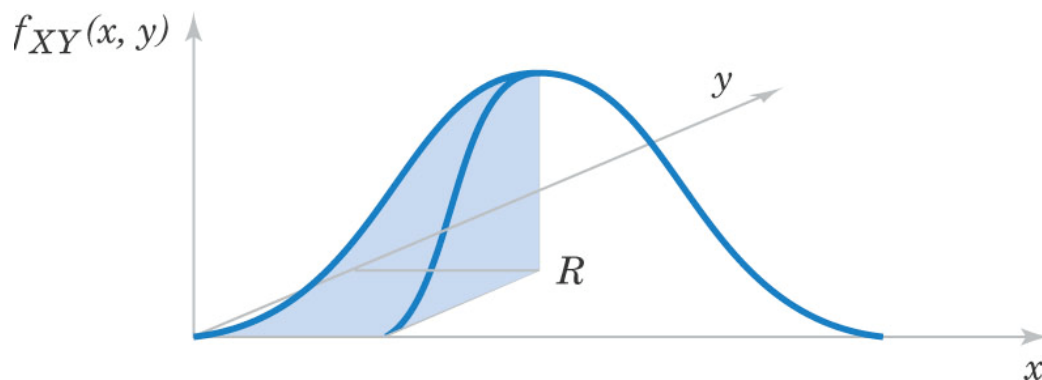
# Joint Probability Density Function Defined

The **joint probability density function** for the continuous random variables  $X$  and  $Y$ , denoted as  $f_{XY}(x,y)$ , satisfies the following properties:

(1)  $f_{XY}(x,y) \geq 0$  for all  $x, y$

(2) 
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) dx dy = 1$$

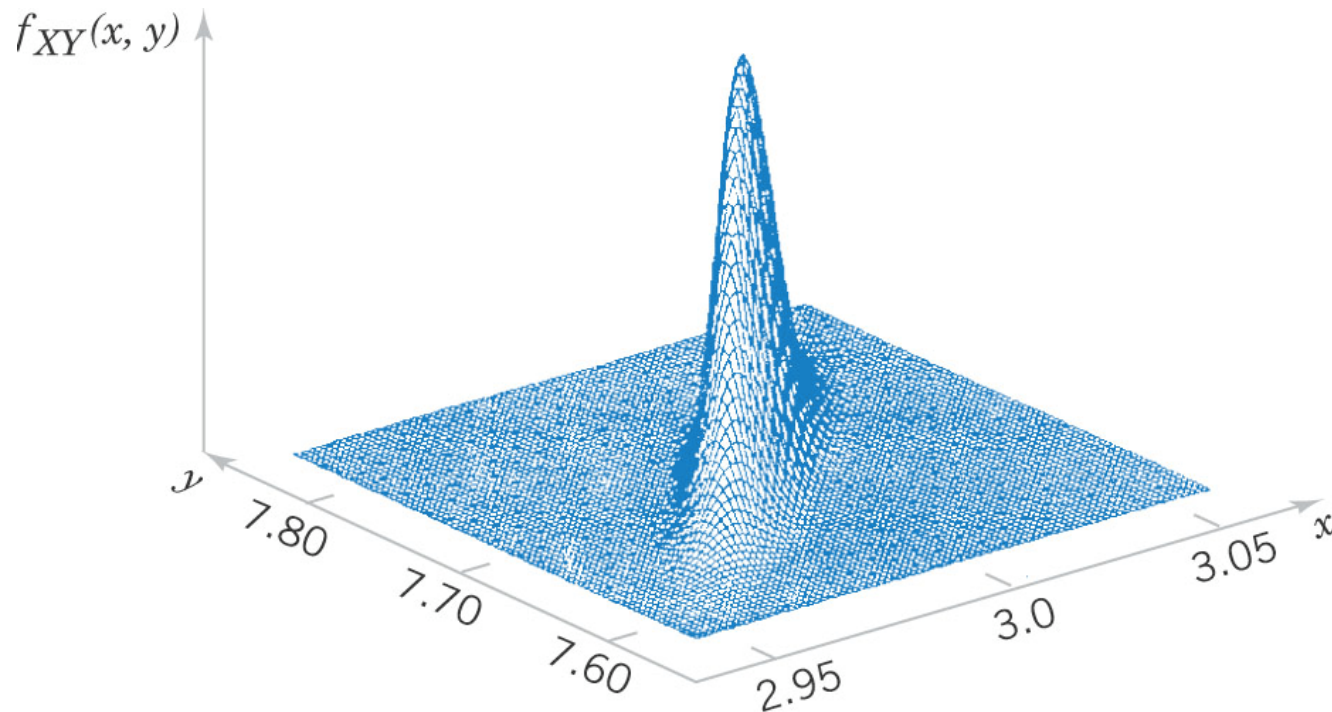
(3) 
$$P((X,Y) \subset R) = \iint_R f_{XY}(x,y) dx dy \quad (5-2)$$



**Figure 5-2** Joint probability density function for the random variables  $X$  and  $Y$ . Probability that  $(X, Y)$  is in the region  $R$  is determined by the **volume** of  $f_{XY}(x,y)$  over the region  $R$ .



# Joint Probability Density Function Graph



**Figure 5-3** Joint probability density function for the continuous random variables  $X$  and  $Y$  of expression levels of two different genes. Note the asymmetric, narrow ridge shape of the PDF – indicating that small values in the  $X$  dimension are more likely to occur when small values in the  $Y$  dimension occur.

# Marginal Probability Distributions (continuous)

- Rather than summing a discrete joint PMF, we integrate a continuous joint PDF.
- The marginal PDFs are used to make probability statements about one variable.
- If the joint probability density function of random variables  $X$  and  $Y$  is  $f_{XY}(x,y)$ , the marginal probability density functions of  $X$  and  $Y$  are:

$$f_X(x) = \int_y f_{XY}(x, y) dy$$

$$f_Y(y) = \int_x f_{XY}(x, y) dx \quad (5-3)$$

$$f_X(x) = \sum_y f_{XY}(x, y)$$

$$f_Y(y) = \sum_x f_{XY}(x, y)$$

# Conditional Probability Density Function Defined

Given continuous random variables  $X$  and  $Y$  with joint probability density function  $f_{XY}(x, y)$ , the conditional probability density function of  $Y$  given  $X=x$  is

$$f_{Y|x}(y) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{f_{XY}(x, y)}{\int_y f_{XY}(x, y) dy} \text{ if } f_X(x) > 0 \quad (5-4)$$

which satisfies the following properties:

(1)  $f_{Y|x}(y) \geq 0$

(2)  $\int f_{Y|x}(y) dy = 1$

(3)  $P(Y \in B | X = x) = \int_B f_{Y|x}(y) dy$  for any set  $B$  in the range of  $Y$

Compare to discrete:  $P(Y=y | X=x) = f_{XY}(x, y) / f_X(x)$

# Conditional Probability Distributions

- Conditional probability distributions can be developed for multiple random variables by extension of the ideas used for two random variables.
- Suppose  $p = 5$  and we wish to find the distribution of  $X_1, X_2$  and  $X_3$  conditional on  $X_4=x_4$  and  $X_5=x_5$ .

$$f_{X_1 X_2 X_3 | x_4 x_5}(x_1, x_2, x_3) = \frac{f_{X_1 X_2 X_3 X_4 X_5}(x_1, x_2, x_3, x_4, x_5)}{f_{X_4 X_5}(x_4, x_5)}$$

for  $f_{X_4 X_5}(x_4, x_5) > 0$ .



# Independence for Continuous Random Variables

For random variables  $X$  and  $Y$ , if any one of the following properties is true, the others are also true. Then  $X$  and  $Y$  are **independent**.

(1)  $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$

(2)  $f_{Y|x}(y) = f_Y(y)$  for all  $x$  and  $y$  with  $f_X(x) > 0$

(3)  $f_{X|y}(x) = f_X(x)$  for all  $x$  and  $y$  with  $f_Y(y) > 0$

(4)  $P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$  for any sets  $A$  and  $B$  in the range of  $X$  and  $Y$ , respectively. (5–7)

$P(Y=y|X=x)=P(Y=y)$  **for any  $x$**  or

$P(X=x|Y=y)=P(X=x)$  **for any  $y$**  or

$P(X=x, Y=y)=P(X=x) \cdot P(Y=y)$  **for any  $x$  and  $y$**

# Covariation, Correlations

Quick and dirty check for  
**linear** (in)dependence  
between variables

# Covariance Defined

Covariance is a number quantifying the average *linear* dependence between two random variables.

The covariance between the random variables  $X$  and  $Y$ , denoted as  $\text{cov}(X, Y)$  or  $\sigma_{XY}$  is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X\mu_Y$$

Montgomery, Runger 5<sup>th</sup> edition Eq. (5–14)

The units of  $\sigma_{XY}$  are the units of  $X$  times the units of  $Y$ .

Unlike the range of the variance, covariance can be negative:  $-\infty < \sigma_{XY} < \infty$ .



Covariance - 1 number to measure dependance between random variables

$\text{Cov}(X, Y)$  or  $\sigma_{xy}$

$$\begin{aligned}\sigma_{xy} &= E[(X - \mu_x) \cdot (Y - \mu_y)] = \\ &= E(X \cdot Y) - \mu_x \cdot \mu_y\end{aligned}$$

- $\text{Var}(X) = \text{Cov}(X, X)$
- If  $X$  &  $Y$  are independent

$$\text{Cov}(X, Y) = E[X - \mu_x] \cdot E[Y - \mu_y] = 0$$

- $-\infty$  <  $\text{Cov}(X, Y)$  <  $+\infty$  Can be negative!

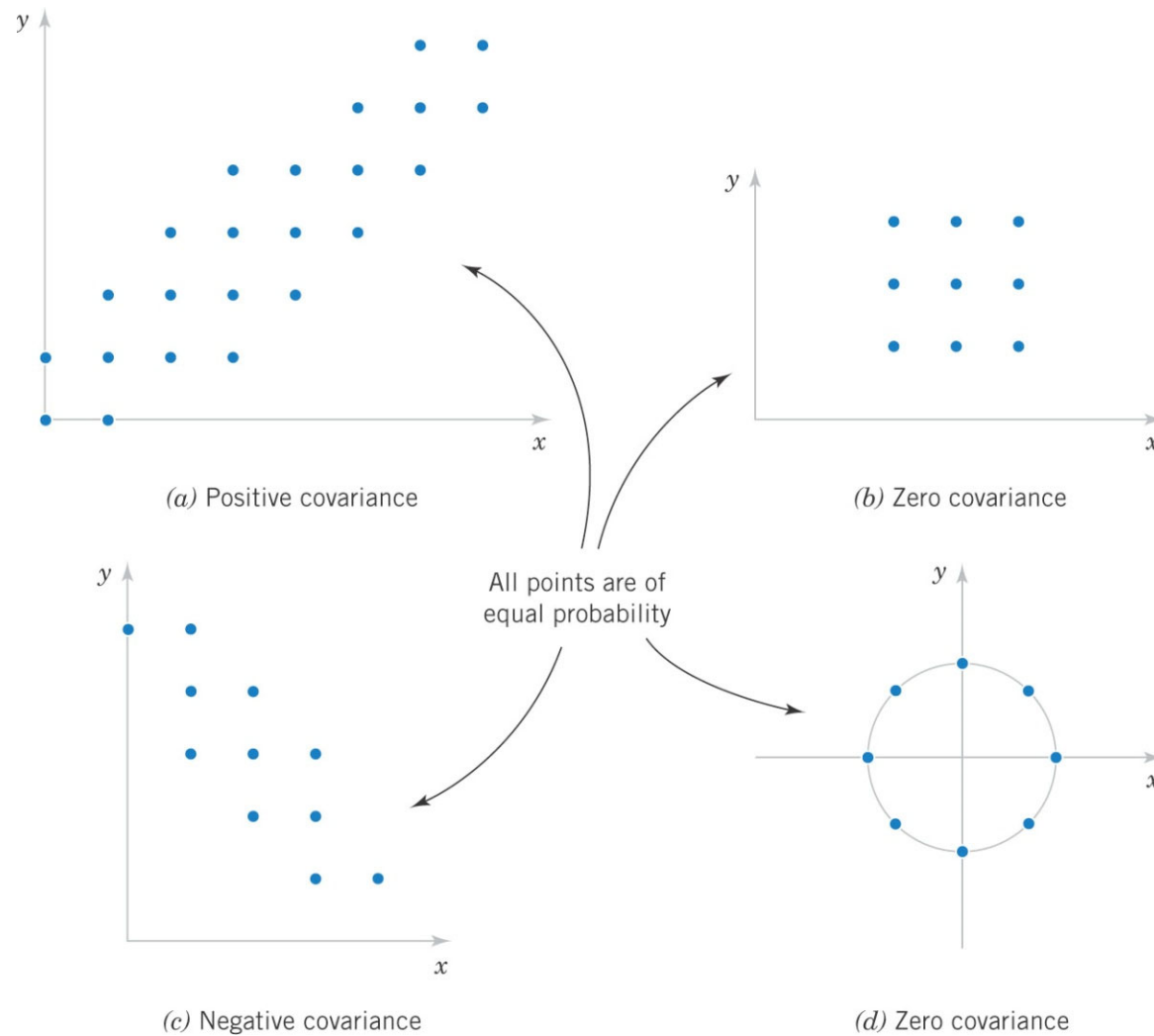
# Covariance and PMF tables

y = number of times city name is stated	x = number of bars of signal strength		
	1	2	3
1	0.01	0.02	0.25
2	0.02	0.03	0.20
3	0.02	0.10	0.05
4	0.15	0.10	0.05

The probability distribution of Example 5-1 is shown.

By inspection, note that the **larger probabilities** occur as  $X$  and  $Y$  move in opposite directions. This indicates a **negative covariance**.

# Covariance and Scatter Patterns



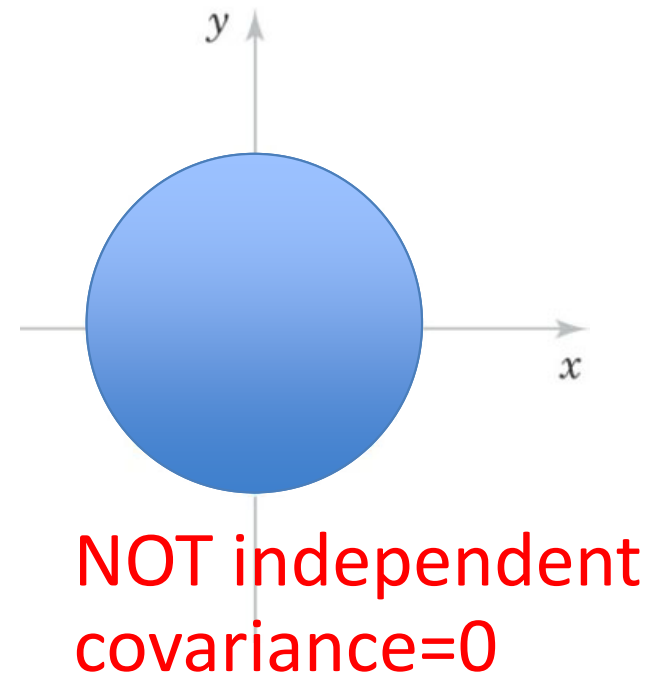
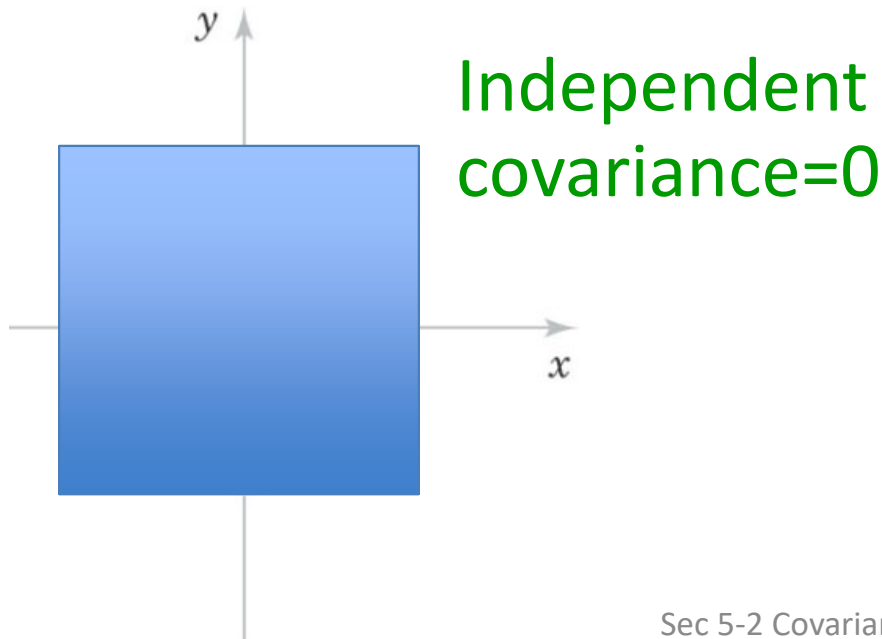
**Figure 5-13** Joint probability distributions and the sign of  $\text{cov}(X, Y)$ . Note that covariance is a measure of linear relationship. Variables with non-zero covariance are **correlated**.

# Independence Implies $\sigma = \rho = 0$ but not vice versa

- If  $X$  and  $Y$  are independent random variables,

$$\sigma_{XY} = \rho_{XY} = 0 \quad (5-17)$$

- $\rho_{XY} = 0$  is necessary, but **not a sufficient** condition for independence.





# Correlation is “normalized covariance”

- Also called:  
**Pearson correlation coefficient**

$\rho_{XY} = \sigma_{XY} / \sigma_X \sigma_Y$   
is the covariance  
normalized to  
be  $-1 \leq \rho_{XY} \leq 1$



Karl Pearson (1852– 1936)  
English mathematician and biostatistician



Prove that  $\rho_{xy}$  is in  $[-1, 1]$

$$Z_x = \frac{X - \mu_x}{\sigma_x}; \quad Z_y = \frac{Y - \mu_y}{\sigma_y}$$

$$0 \leq E((Z_x - Z_y)^2) = E(Z_x^2) + E(Z_y^2) - 2E(Z_x \cdot Z_y) = 2 - 2 \frac{1}{\sigma_x \sigma_y} E((X - \mu_x)(Y - \mu_y)) =$$

$$2 - 2\rho_{xy} \implies \boxed{\rho_{xy} \leq 1}$$

$$0 \leq E((Z_x + Z_y)^2) = E(Z_x^2) + E(Z_y^2) + 2E(Z_x \cdot Z_y) = 2 + 2\rho_{xy} \implies$$

$$\implies \boxed{\rho_{xy} \geq -1}$$

# Spearman rank correlation

- **Pearson correlation** tests for **linear relationship** between X and Y
- **Unlikely for** variables with **broad distributions** → non-linear effects dominate
- **Spearman correlation** tests for any **monotonic relationship** between X and Y
- **Calculate ranks** (1 to n),  $r_X(i)$  and  $r_Y(i)$  of variables in both samples. Calculate Pearson correlation between ranks:  
 **$Spearman(X,Y) = Pearson(r_X, r_Y)$**
- **Ties:** convert to fractions, e.g. tie for 6s and 7s place both get 6.5. This can lead to artefacts.
- If lots of ties: use **Kendall rank correlation** (Kendall tau)



Credit: XKCD  
comics

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*Example 3.10.* Let an experiment consist of drawing a card at random from a standard deck of 52 playing cards. Define events  $A$  and  $B$  as “the card is a ♠” and “the card is a queen.” Are the events  $A$  and  $B$  independent? By definition,  $P(A \cdot B) = P(Q \spadesuit) = \frac{1}{52}$ . This is the product of  $P(\spadesuit) = \frac{13}{52}$  and  $P(Q) = \frac{4}{52}$ , and events  $A$  and  $B$  in question are independent. In this situation, intuition provides no help. Now, pretend that the  $2\heartsuit$  is drawn and excluded from the deck prior to the experiment. Events  $A$  and  $B$  become dependent since

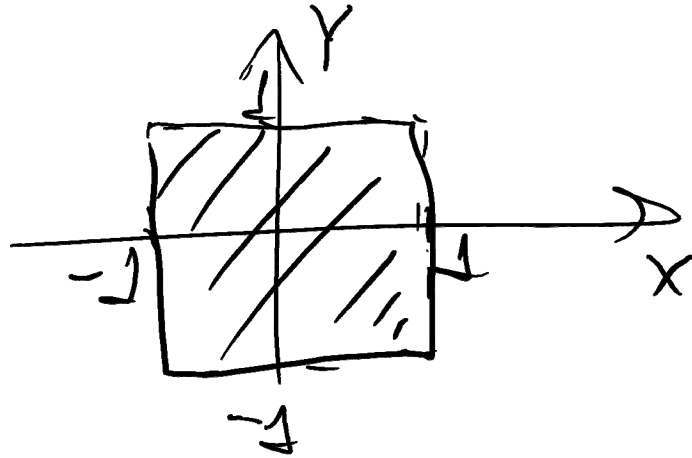
$$\mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{13}{51} \cdot \frac{4}{51} \neq \frac{1}{51} = \mathbb{P}(A \cdot B).$$



Example 1:

Uniform distribution in the square

$$-1 < X < 1, \quad -1 < Y < 1$$



$$\begin{cases} f_{XY}(x, y) = c & \text{if } -1 < x < 1 \text{ and } -1 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \int_{\text{square}} dx dy f_{XY}(x, y) = c \cdot \text{Area} = c \cdot 4 \rightarrow c = \frac{1}{4}$$



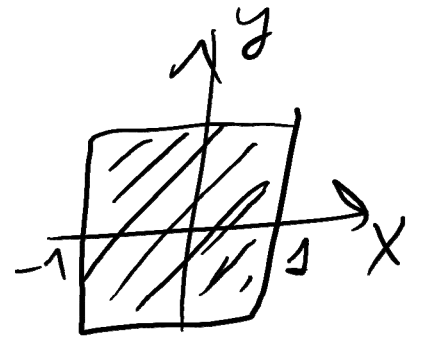


Are  $X$  and  $Y$  independent? Yes they are

Let's test if  $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy =$$

$$= \int_{-1}^1 \frac{1}{4} dy = \frac{1}{2} \text{ if } -1 < x < 1$$

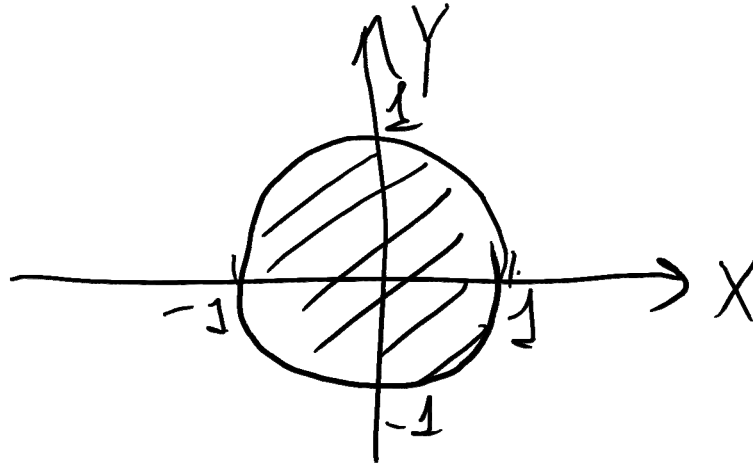


Same for  $f_Y(y) = \frac{1}{2}$  if  $-1 < y < 1$

$$\frac{1}{4} = f_{XY}(x, y) = \frac{1}{2} \cdot \frac{1}{2} = f_X(x) \cdot f_Y(y)$$

0 otherwise if both  $x$  &  $y$  are in  $[-1, 1]$

X and Y are uniformly distributed in  
the disc  $x^2+y^2\leq 1$



Are they independent?

A. yes

B. no

C. I could not figure it out

Get your i-clickers

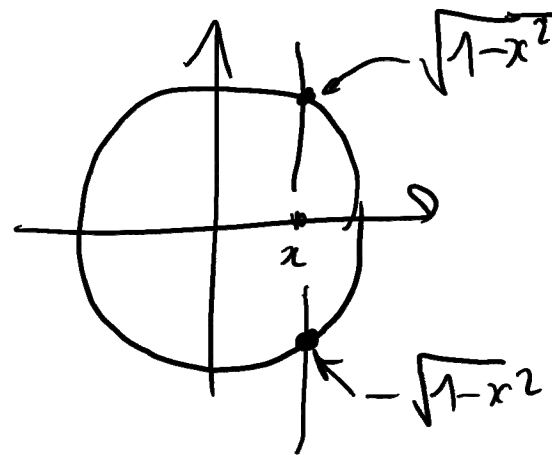


Joint PDF  $f_{XY}(x, y) = \frac{1}{\text{area}} = \frac{1}{\pi}$  if  $x, y$  in the disc

Marginal distributions: 0 - otherwise

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{dy}{\pi} = \frac{2\sqrt{1-x^2}}{\pi}$$

Same for  $f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi}$



$$\frac{1}{\pi} = f_{XY}(x, y) \neq \frac{2}{\pi} \sqrt{1-x^2} \cdot \frac{2}{\pi} \sqrt{1-y^2} = f_X(x) \cdot f_Y(y)$$

Variables are NOT independent

Credit: XKCD  
comics

# WHY ARE THERE SLAVES IN THE BIBLE

WHY DO TWINS HAVE DIFFERENT FINGERPRINTS  
WHY ARE AMERICANS AFRAID OF DRAGONS

WHY IS HTTPS CROSSED OUT IN RED  
WHY IS THERE A LINE THROUGH HTTPS  
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK  
WHY IS HTTPS IMPORTANT

# QUESTIONS FOUND IN GOOGLE AUTOCOMPLETE



WHY ARE THERE WEEKS  
WHY DO I FEEL DIZZY

WHY DO WHALES JUMP  
WHY ARE WITCHES GREEN  
WHY ARE THERE MIRRORS ABOVE BEDS  
WHY DO I SAY UH  
WHY IS SEA SALT BETTER  
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WHY IS THERE NOT A POKEMON MMO  
WHY IS THERE LAUGHING IN TV SHOWS  
WHY ARE THERE DOORS ON THE FREEWAY  
WHY ARE THERE SO MANY SVCHOST.EXE RUNNING  
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA  
WHY ARE THERE SCARY SOUNDS IN MINECRAFT  
WHY IS THERE KICKING IN MY STOMACH  
WHY ARE THERE TWO SLASHES AFTER HTTP  
WHY ARE THERE CELEBRITIES  
WHY DO SNAKES EXIST  
WHY DO OYSTERS HAVE PEARLS  
WHY ARE DUCKS CALLED DUCKS  
WHY DO THEY CALL IT THE CLAP  
WHY ARE KYLE AND CARTMAN FRIENDS  
WHY IS THERE AN ARROW ON AANG'S HEAD  
WHY ARE TEXT MESSAGES BLUE  
WHY ARE THERE MUSTACHES ON CLOTHES  
WHY ARE THERE MUSTACHES ON CARS  
WHY ARE THERE MUSTACHES EVERYWHERE  
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WHY IS THERE SO MUCH RAIN IN OHIO  
WHY IS OHIO WEATHER SO WEIRD

WHY DO IGUANAS DIE

WHY AREN'T ECONOMISTS RICH  
WHY DO AMERICANS CALL IT SOCCER  
WHY ARE MY EARS RINGING  
WHY ARE THERE SO MANY AVENGERS  
WHY ARE THE AVENGERS FIGHTING THE X MEN  
WHY IS WOLVERINE NOT IN THE AVENGERS

WHY ARE THERE DINOSAUR GHOSTS  
WHY IS EARTH TILTED  
WHY IS SPACE BLACK  
WHY IS OUTER SPACE SO COLD  
WHY ARE THERE PYRAMIDS ON THE MOON  
WHY IS NASA SHUTTING DOWN



WHY IS THERE AN OWL IN MY BACKYARD  
WHY IS THERE AN OWL OUTSIDE MY WINDOW  
WHY IS THERE AN OWL ON THE DOLLAR BILL  
WHY DO OWLS ATTACK PEOPLE  
WHY ARE AK 47s SO EXPENSIVE  
WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE  
WHY ARE THERE GODS  
WHY ARE THERE TWO SPOCKS

WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN  
WHY IS PSYCHIC WEAK TO BUG  
WHY DO CHILDREN GET CANCER  
WHY IS POSEIDON ANGRY WITH ODYSSEUS  
WHY IS THERE ICE IN SPACE



WHY ARE THERE BRIDESMAIDS  
WHY DO DYING PEOPLE REACH UP  
WHY AREN'T THERE VARICOSE ARTERIES  
WHY ARE OLD KUNGONS DIFFERENT  
WHY ARE THERE TINY SPIDERS IN MY HOUSE  
WHY DO SPIDERS COME INSIDE  
WHY ARE THERE HUGE SPIDERS IN MY HOUSE  
WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE  
WHY ARE THERE SPIDERS IN MY ROOM  
WHY ARE THERE SO MANY SPIDERS IN MY ROOM  
WHY DO SPIDER BITES ITCH  
WHY IS DYING SO SCARY

WHY ARE THERE FEMALE MR NIMES



WHY IS MT VESUVIUS THERE  
WHY DO THEY SAY T MINUS  
WHY ARE THERE OBELISKS  
WHY ARE WRESTLERS ALWAYS WET  
WHY ARE OCEANS BECOMING MORE ACIDIC  
WHY IS ARWEN DYING  
WHY AREN'T MY QUAIL LAYING EGGS  
WHY AREN'T MY QUAIL EGGS HATCHING  
WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA

WHY IS LIFE SO BORING  
WHY ARE CIGARETTES LEGAL  
WHY ARE THERE DUCKS IN MY POOL  
WHY IS JESUS WHITE  
WHY IS THERE LIQUID IN MY EAR  
WHY DO Q TIPS FEEL GOOD  
WHY DO GOOD PEOPLE DIE  
WHY AREN'T THERE GUNS IN HARRY POTTER  
WHY ARE ULTRASOUNDS IMPORTANT  
WHY ARE ULTRASOUND MACHINES EXPENSIVE  
WHY IS STEALING WRONG

WHY IS PROGRAMMING SO HARD  
WHY IS THERE A 0 OHM RESISTOR  
WHY DO AMERICANS HATE SOCCER  
WHY DO RHYMES SOUND GOOD  
WHY DO TREES DIE  
WHY IS THERE NO SOUND ON CNN  
WHY AREN'T POKEMON REAL  
WHY AREN'T BULLETS SHARP  
WHY DO DREAMS SEEM SO REAL

WHY IS THERE NO GPS IN LAPTOPS  
WHY DO KNEES CLICK  
WHY AREN'T THERE E GRADES  
WHY IS ISOLATION BAD  
WHY DO BOYS LIKE ME  
WHY DON'T BOYS LIKE ME  
WHY IS THERE ALWAYS A JAVA UPDATE  
WHY ARE THERE RED DOTS ON MY THIGHS  
WHY IS LYING GOOD

WHY ARE DOGS AFRAID OF FIREWORKS  
WHY IS THERE NO KING IN ENGLAND



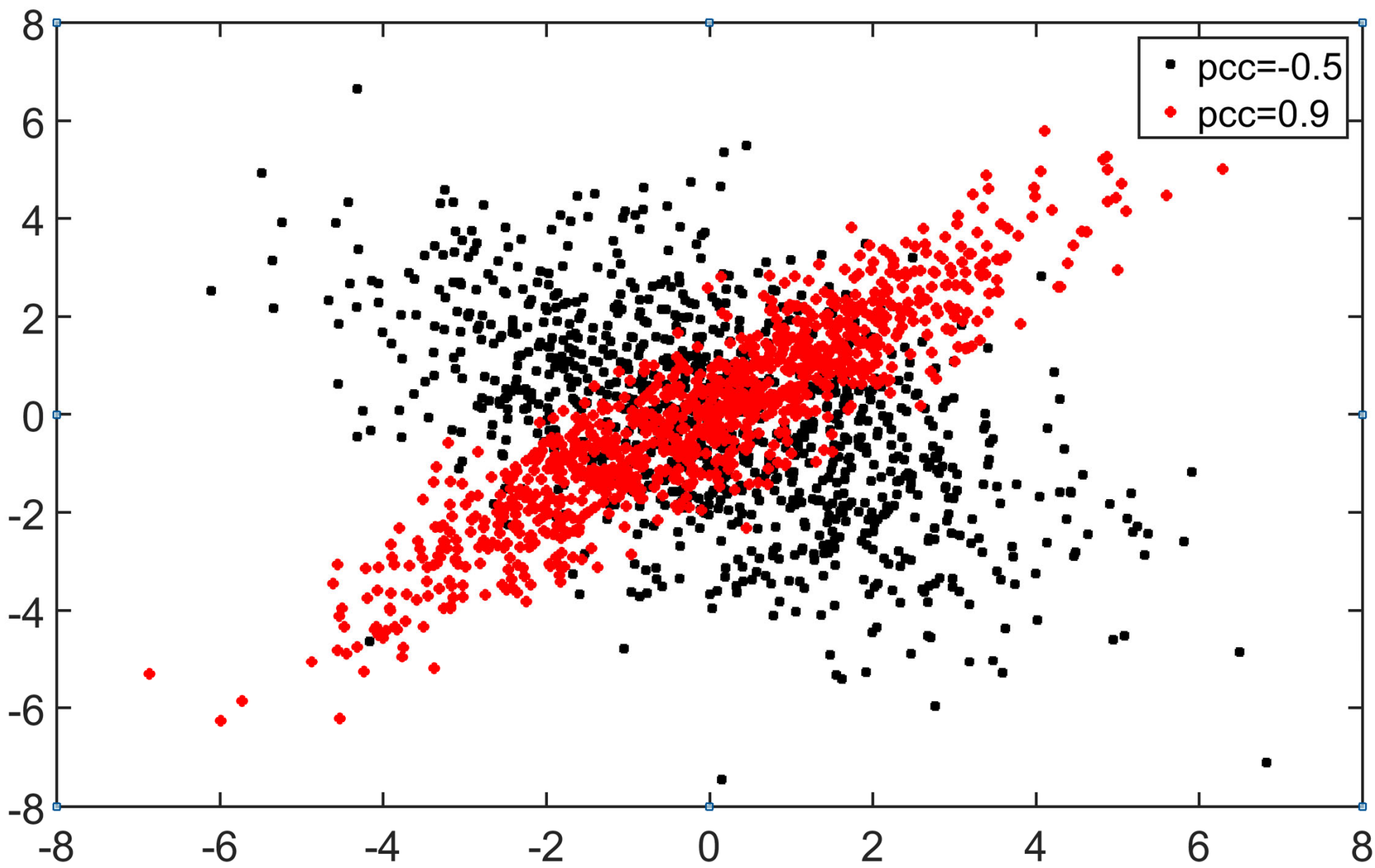
# Matlab exercise: Correlation/Covariation

- Generate a sample with **Stats=100,000** of two Gaussian random variables **r1** and **r2** which have **mean 0** and **standard deviation 2** and are:
  - **Uncorrelated**
  - Correlated with **correlation coefficient 0.9**
  - Correlated with **correlation coefficient -0.5**
  - Trick: first make **uncorrelated r1** and **r2**. Then make a new variable:  **$r1_{mix} = mix \cdot r2 + (1 - mix^2)^{0.5} \cdot r1$** ; where **mix = corr. coeff.**
- For each value of **mix** calculate covariance and **correlation coefficient** between **r1mix** and **r2**
- In each case make a scatter plot: **`plot(r1mix,r2,'k.')`**;

# Matlab exercise: Correlation/Covariation

1. Stats=100000;
2. r1=2.\*randn(Stats,1);
3. r2=2.\*randn(Stats,1);
4. disp('Covariance matrix='); disp(cov(r1,r2));
5. disp('Correlation=');disp(corr(r1,r2));
6. figure; plot(r1,r2,'k.');
7. mix=0.9; **%Mixes r2 to r1 but keeps same variance**
8. r1mix=mix.\*r2+sqrt(1-mix.^2).\*r1;
9. disp('Covariance matrix='); disp(cov(r1mix,r2));
10. disp('Correlation=');disp(corr(r1mix,r2));
11. figure; plot(r1mix,r2,'k.');
12. mix=-0.5; **%REDO LINES 8-11**





Credit: XKCD  
comics

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