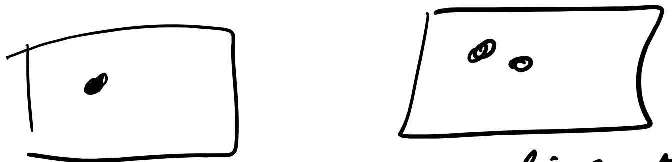
Inductive probability relies on combinatorics or the art of counting combinations

How many ways to choose a sample of $K$ objects out of a population of $n$ objects

| | Order matters | Order does not matter |
|---|---|---|
| Replace | $n \times n \times n \times \ldots n = n^K$ | ~~$\frac{n^K}{K!}$~~ not all objects are different |
| Do not replace | $n \times (n-1) \times (n-2) \times \ldots \times (n-K+1) = \frac{n!}{(n-K)!}$ | All objects are different → $\frac{n!}{(n-K)!} \times \frac{1}{K!} = \binom{n}{K}$ |

How to solve the problem of K out of n with replacement but where order does not matter?

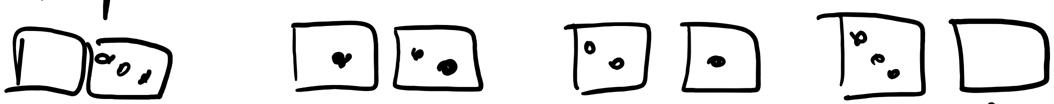
Let's solve n=2 problem first:



object 1 object 2

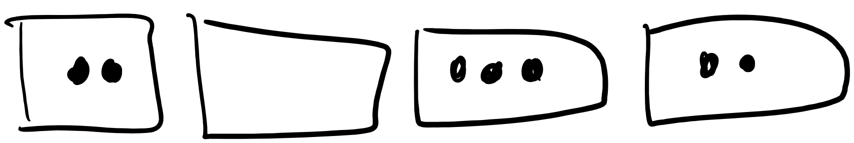
K=3

4 possibilities

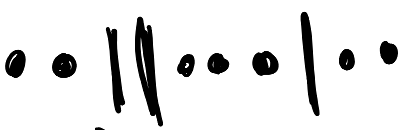


(1) (2) (3) (4)

---



n=4, k=7



K dots, n-1 box boundaries

$$\binom{k+n-1}{k} = \frac{(k+n-1)!}{k!\,(n-1)!}$$ ways to distribute

# Sampling table

How many ways to choose a sample of k objects out of population of n objects?

| | Order matters | Order does not matter |
|---|---|---|
| Replacement | $(n)^k$ | Difficult: $\binom{n+k-1}{k}$= $\frac{(n+k-1)!}{(n-1)!k!}$ |
| No replacement | $n(n-1)(n-2)..(n-k+1)=$ $=\frac{n!}{(n-k)!}$ | $\binom{n}{k}$=$\frac{n!}{(n-k)!k!}$ |

# Example

- A DNA of 100 bases is characterized by its numbers of 4 nucleotides: $d_A$, $d_C$, $d_G$, and $d_T$ ($d_A+d_C+d_G+d_T=100$)
- **<u>I don't care about the sequence</u>** (only about the total numbers of A,C,G, and T
- How many distinct combinations of $d_A$, $d_C$, $d_G$, and $d_T$ are out there?

# What is n and k in this problem?

A. k=100, n=4
B. k=4, n=100
C. k=100, $n=4^{100}$
D. $k=4^{100}$, n=4
E. I don't know

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# What is n and k in this problem?

A. k=100, n=4
B. k=4, n=100
C. k=100, $n=4^{100}$
D. $k=4^{100}$, n=4
E. I don't know

# Is it sampling with or without replacement & does order matter?

A. with replacement, order matters
B. without replacement, any order
C. with replacement, any order
D. without replacement, order matters
E. I don't know

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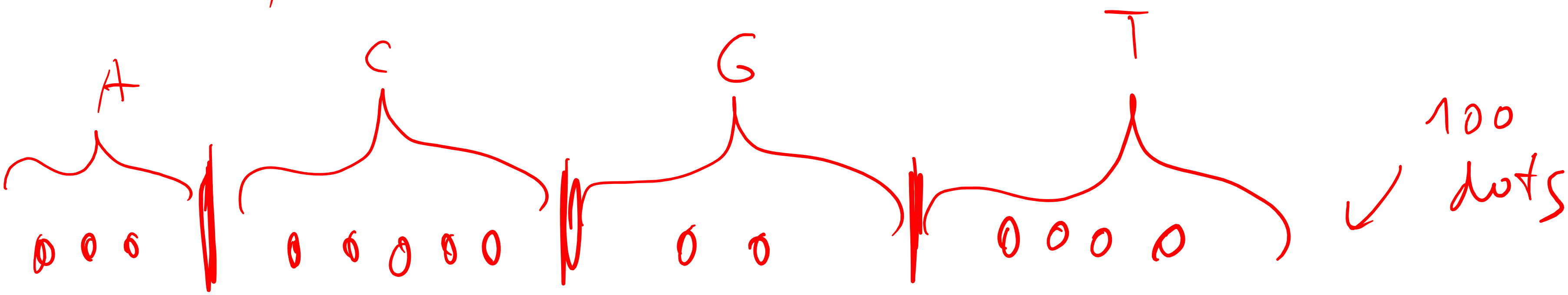
# Is it sampling with or without replacement & does order matter?

A. with replacement, order matters
B. without replacement, any order
C. with replacement, any order
D. without replacement, order matters
E. I don't know

$n = 4, \ k = 100$




$$\# \text{ of possibilities} = \frac{(100+4-1)!}{(4-1)!\ 100!} = \frac{103 \cdot 102 \cdot 101}{3 \cdot 2 \cdot 1} = 176,851$$

If order did matter

$$4^{100} = 2^{200} = 10^{60} \approx \# \text{ atoms in a galaxy}$$

# Probability Axioms, Conditional Probability, Statistical (In)dependence, Circuit Problems

# Axioms of probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If $S$ is the sample space and $E$ is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

(3) For two events $E_1$ and $E_2$ with $E_1 \cap E_2 = \varnothing$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

These axioms imply that:

$$P(\varnothing) = 0$$

$$P(E') = 1 - P(E)$$

if the event $E_1$ is contained in the event $E_2$

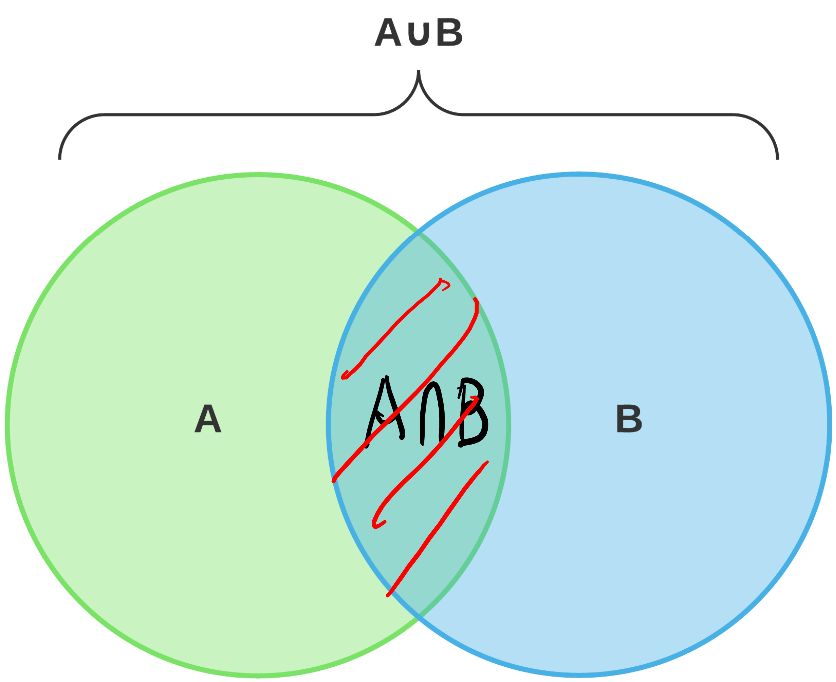
$$P(E_1) \leq P(E_2)$$

# Addition rules following from the Axiom (3)

If $A$ and $B$ are mutually exclusive events, i.e. $A \cap B = \emptyset$

$$P(A \cup B) = P(A) + P(B) \qquad (2\text{-}2)$$

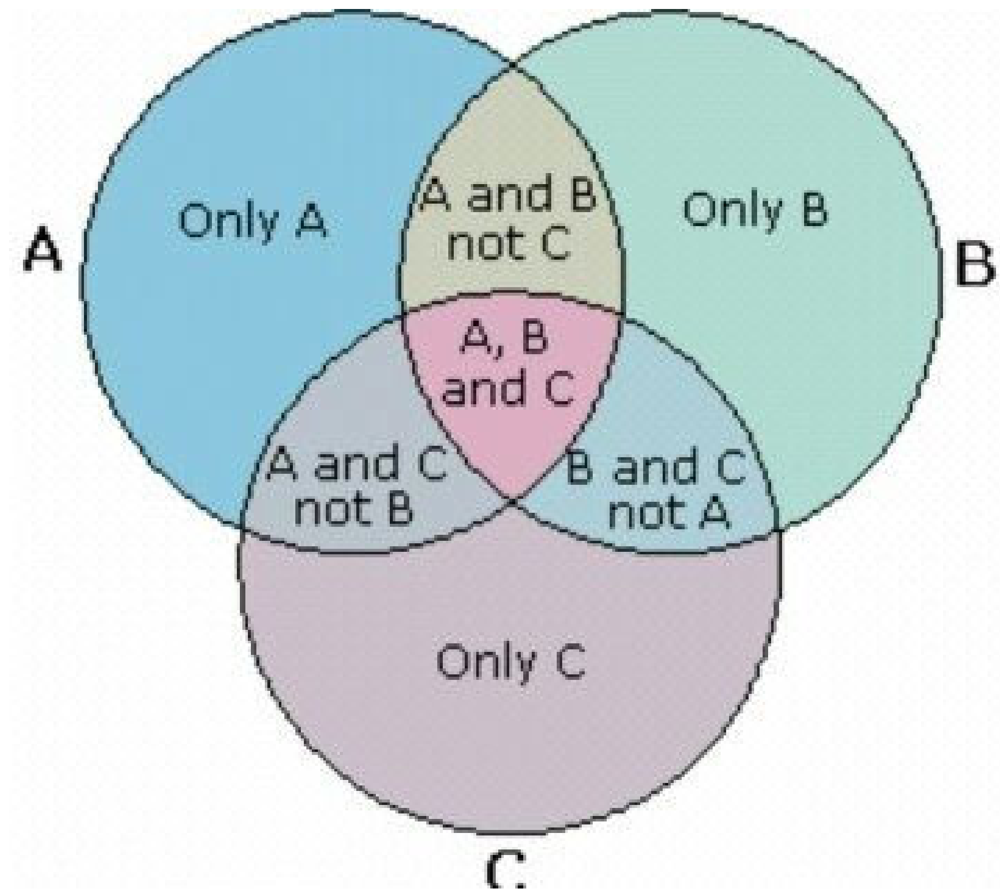
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad (2\text{-}1)$$



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) -$$

$$- P(A \cap B) - P(A \cap C) - P(B \cap C) +$$

$$+ P(A \cap B \cap C).$$



# Conditional probability

The **conditional probability** of an event $B$ given an event $A$, denoted as $P(B|A)$, is

$$P(B|A) = P(A \cap B)/P(A)$$

for $P(A) > 0$.

This definition can be understood in a special case in which all outcomes of a random experiment are equally likely. If there are $n$ total outcomes,

$$P(A) = (\text{number of outcomes in } A)/n$$

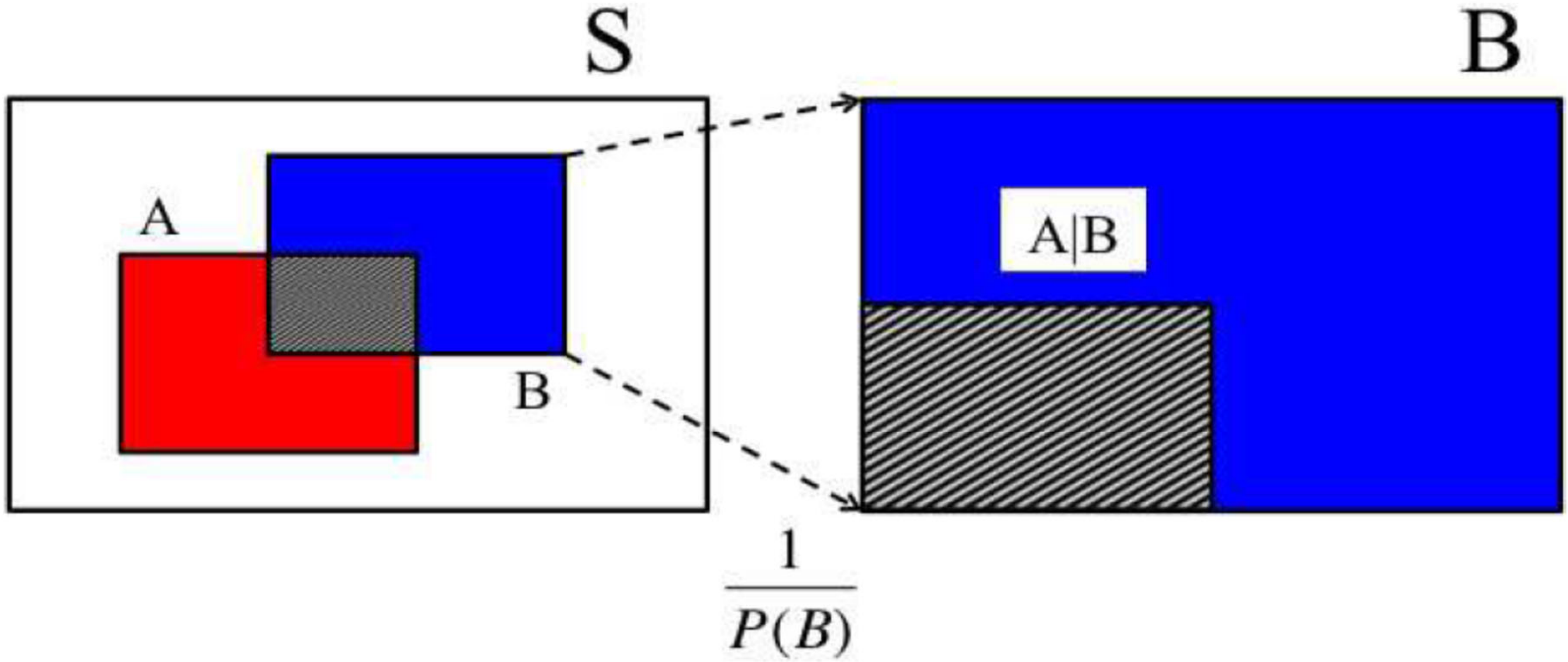
Also,

$$P(A \cap B) = (\text{number of outcomes in } A \cap B)/n$$

Consequently,

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

Therefore, $P(B|A)$ can be interpreted as the relative frequency of event $B$ among the trials that produce an outcome in event $A$.


S
B
A
B
A|B
$\frac{1}{P(B)}$

# Multiplication rule

is just definition of conditional probability

$P(B|A) = P(B \cap A)/P(A) \rightarrow$

$$P(B \cap A) = P(B|A) \cdot P(A)$$

# Drake equation

$$N = R^{*} \cdot f_p \cdot n_e \cdot f_l \cdot f_i \cdot f_c \cdot L$$

- N = The number of civilizations in The Milky Way Galaxy whose electromagnetic emissions are detectable.
- R* = The rate of formation of stars suitable for the development of intelligent life.
- $f_p$ = The fraction of those stars with planetary systems.
- $n_e$ = The number of planets, per solar system, with an environment suitable for life.
- $f_l$ = The fraction of suitable planets on which life actually appears.
- $f_i$ = The fraction of life bearing planets on which intelligent life emerges.
- $f_c$ = The fraction of civilizations that develop a technology that releases detectable signs of their existence into space.
- L = The length of time such civilizations release them

Ms. Perez figures that there is a 5% chance that her company will set up a branch in Phoenix. If it does, she is 10% certain that she will be made its manager. What is the probability that Perez will be a Phoenix branch office manager?

A. 15%
B. 0.5%
C. 50%
D. 5%
E. 10%

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# Statistically independent events

Always true: $P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$

- **Two events**

Two events are **independent** if any one of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

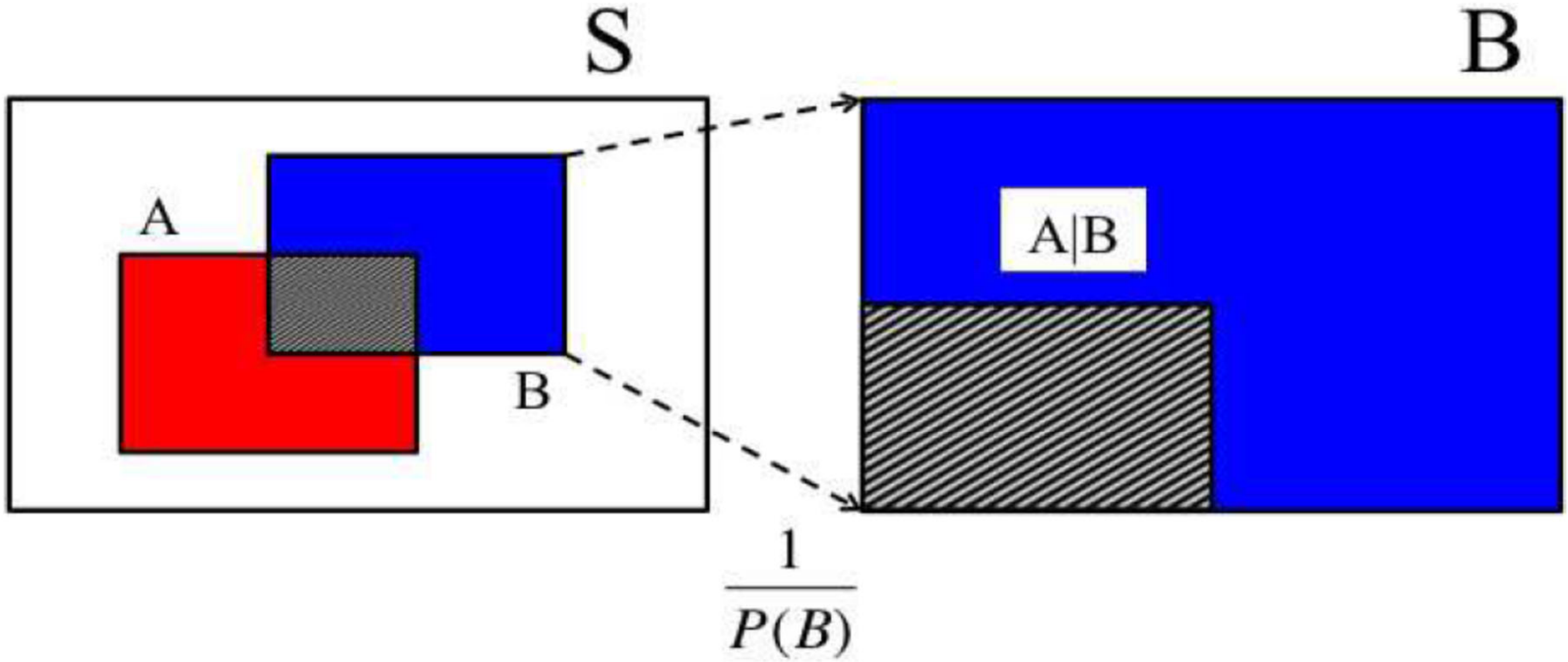
(2) $P(B|A) = P(B)$

(3) $P(A \cap B) = P(A)P(B)$

- **Multiple events**

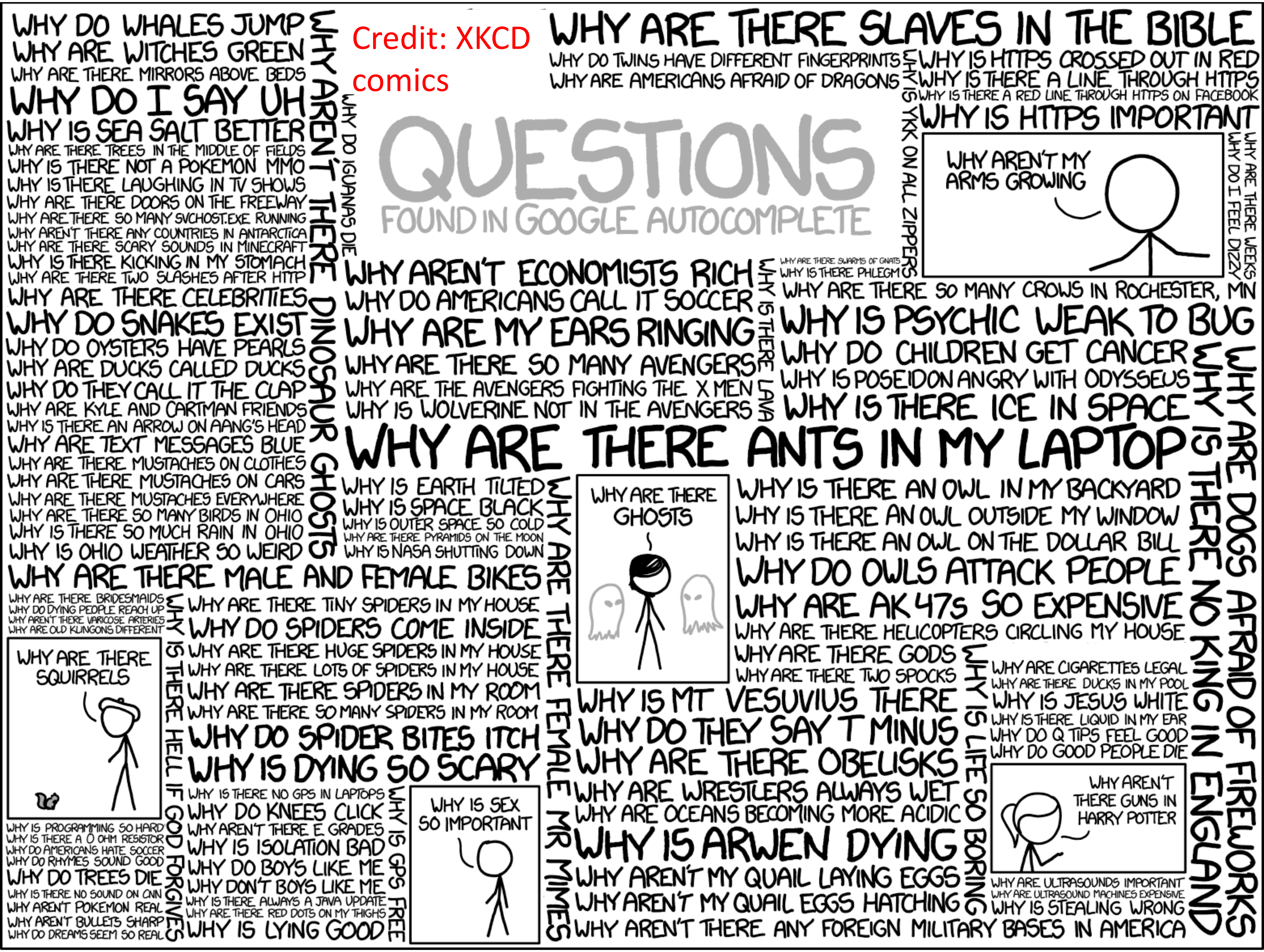
The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots \times P(E_{i_k})$$


S
B
A
B
A|B
$\frac{1}{P(B)}$

*Example 3.10.* Let an experiment consist of drawing a card at random from a standard deck of 52 playing cards. Define events $A$ and $B$ as "the card is a $\spadesuit$" and "the card is a queen." Are the events $A$ and $B$ independent? By definition, $P(A \cdot B) = P(Q\spadesuit) = \frac{1}{52}$. This is the product of $P(\spadesuit) = \frac{13}{52}$ and $P(Q) = \frac{4}{52}$, and events $A$ and $B$ in question are independent. In this situation, intuition provides no help. Now, pretend that the $2\heartsuit$ is drawn and excluded from the deck prior to the experiment. Events $A$ and $B$ become dependent since

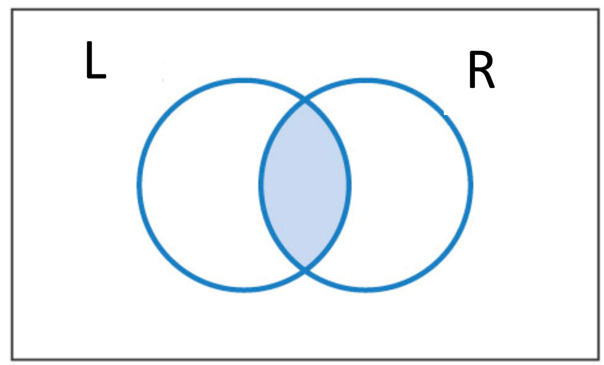
$$\mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{13}{51} \cdot \frac{4}{51} \neq \frac{1}{51} = \mathbb{P}(A \cdot B).$$



Credit: XKCD comics

# Series Circuit

This circuit operates only if there is at least one path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that the devices fail independently. What is the probability that the circuit operates?

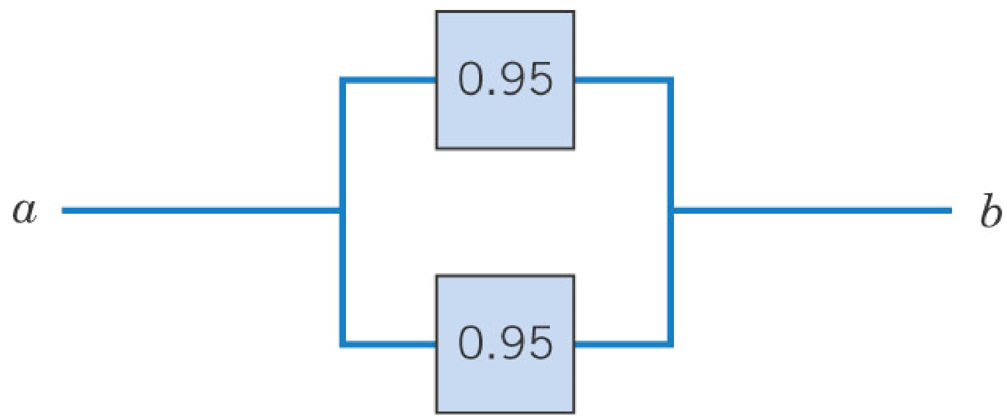



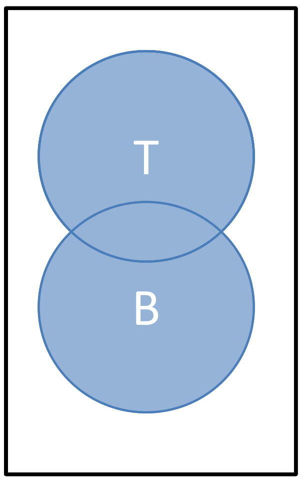

Let L & R denote the events that the left and right devices operate. The probability that the circuit operates is:

$$P(L \text{ and } R) = P(L \cap R) = P(L) * P(R) = 0.8 * 0.9 = 0.72.$$

# Parallel Circuit

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown. Each device fails independently.



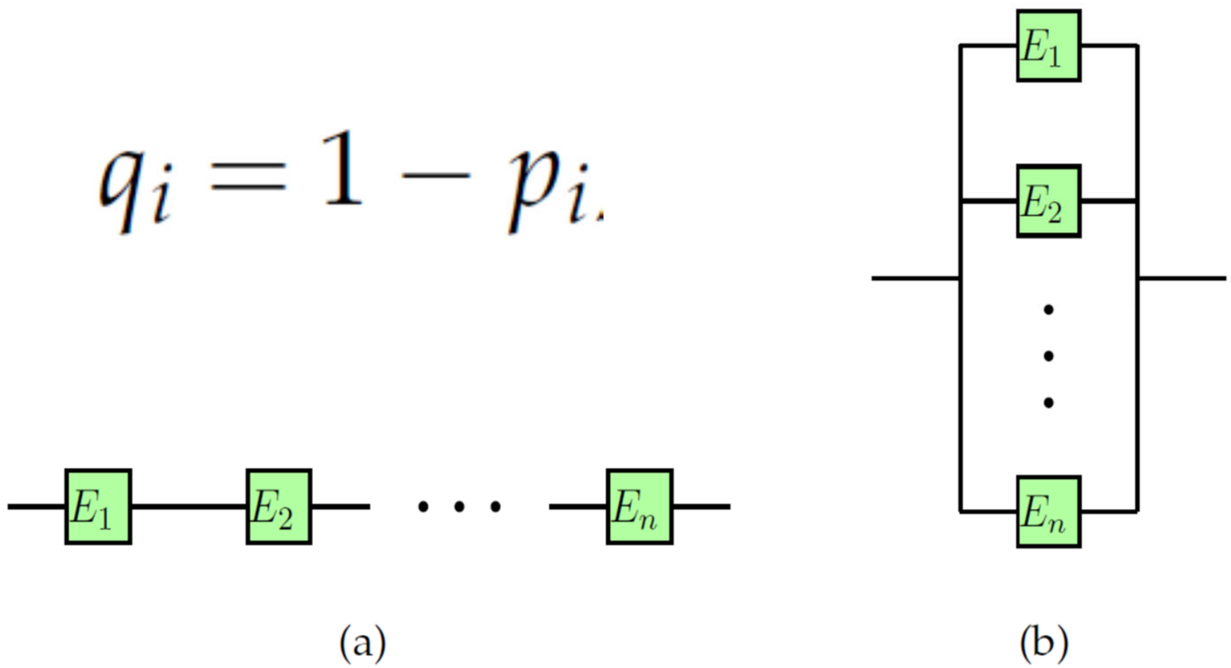




Let T & B denote the events that the top and bottom devices operate. The probability that the circuit operates is:

$P(T \cup B) = 1 - P(T' \cap B') = 1 - P(T')*P(B') = 1 – 0.05^2 = 1 – 0.0025 – 0.9975.$

# Duality between parallel and series circuits

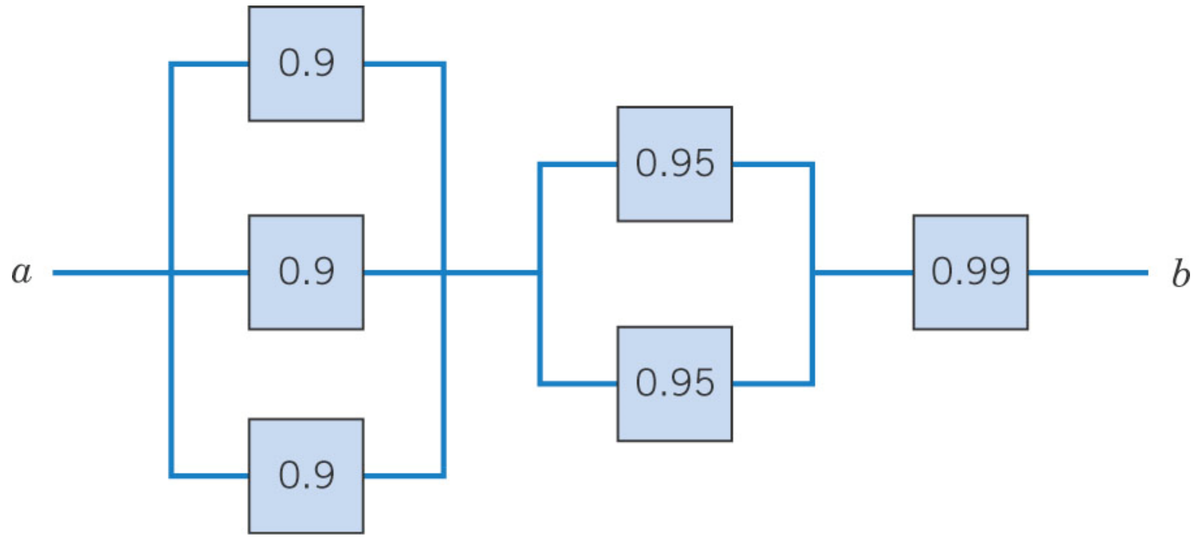
$$q_i = 1 - p_i.$$




| Connection | Notation | Works with prob | Fails with prob |
|---|---|---|---|
| Serial | $E_1 \cap E_2 \cap \cdots \cap E_n$ | $p_1 p_2 \ldots p_n$ | $1 - p_1 p_2 \ldots p_n$ |
| Parallel | $E_1 \cup E_2 \cup \cdots \cup E_n$ | $1 - q_1 q_2 \ldots q_n$ | $q_1 q_2 \ldots q_n$ |

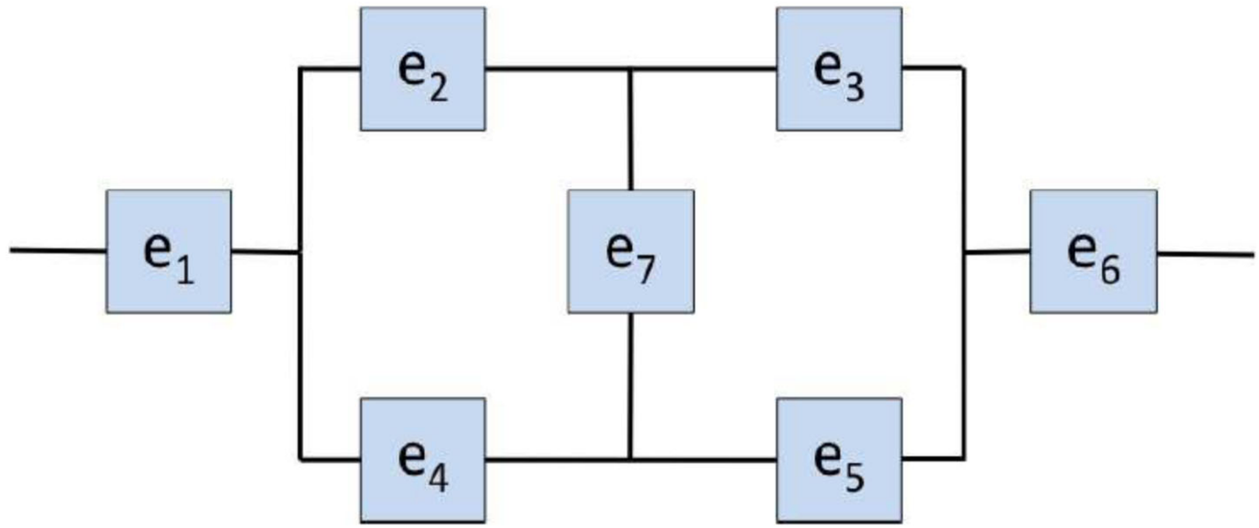
# Advanced Circuit

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown. Each device fails independently.

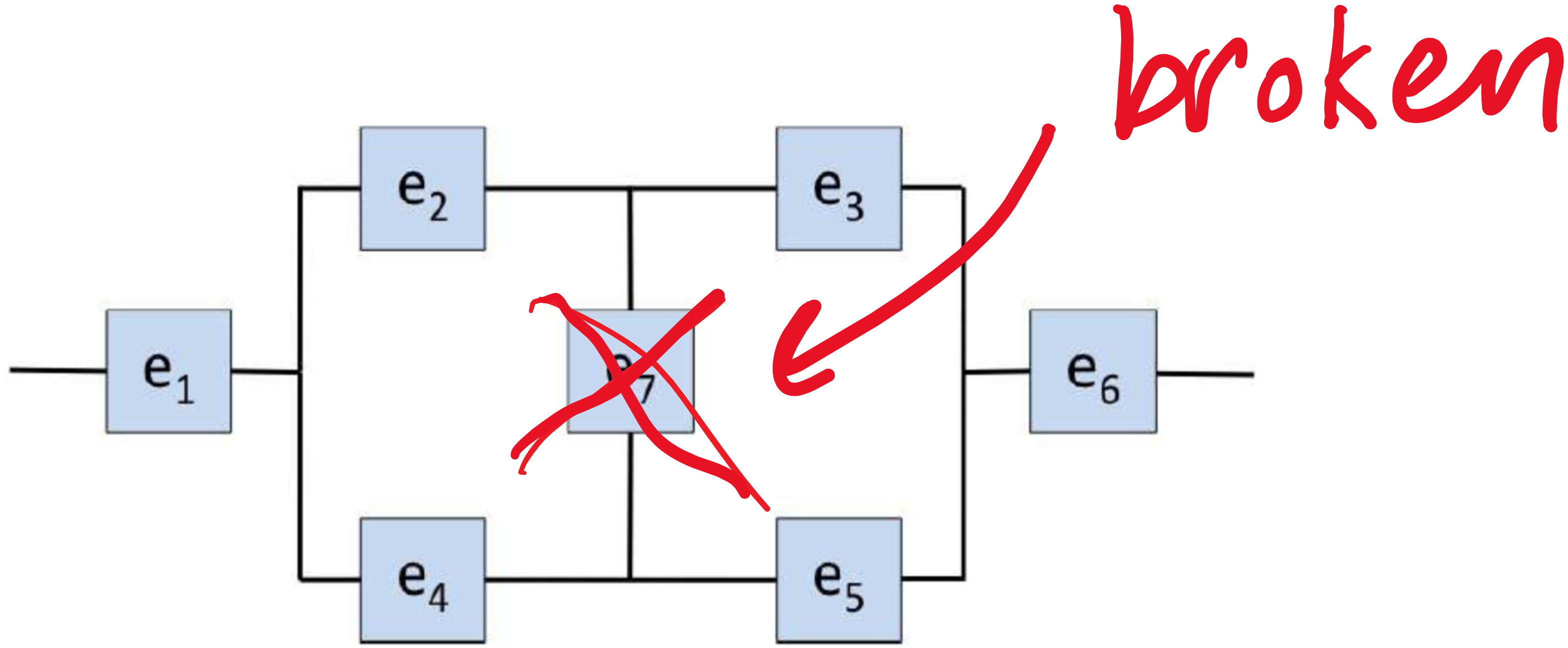



Partition the graph into 3 columns with L & M denoting the left & middle columns.

$P(L) = 1 - 0.1^3$ , and $P(M) = 1 - 0.05^2$, so the probability that the circuit operates is: $(1 - 0.1^3)(1-0.05^2)(0.99) = 0.9875$ (this is a series of parallel circuits).



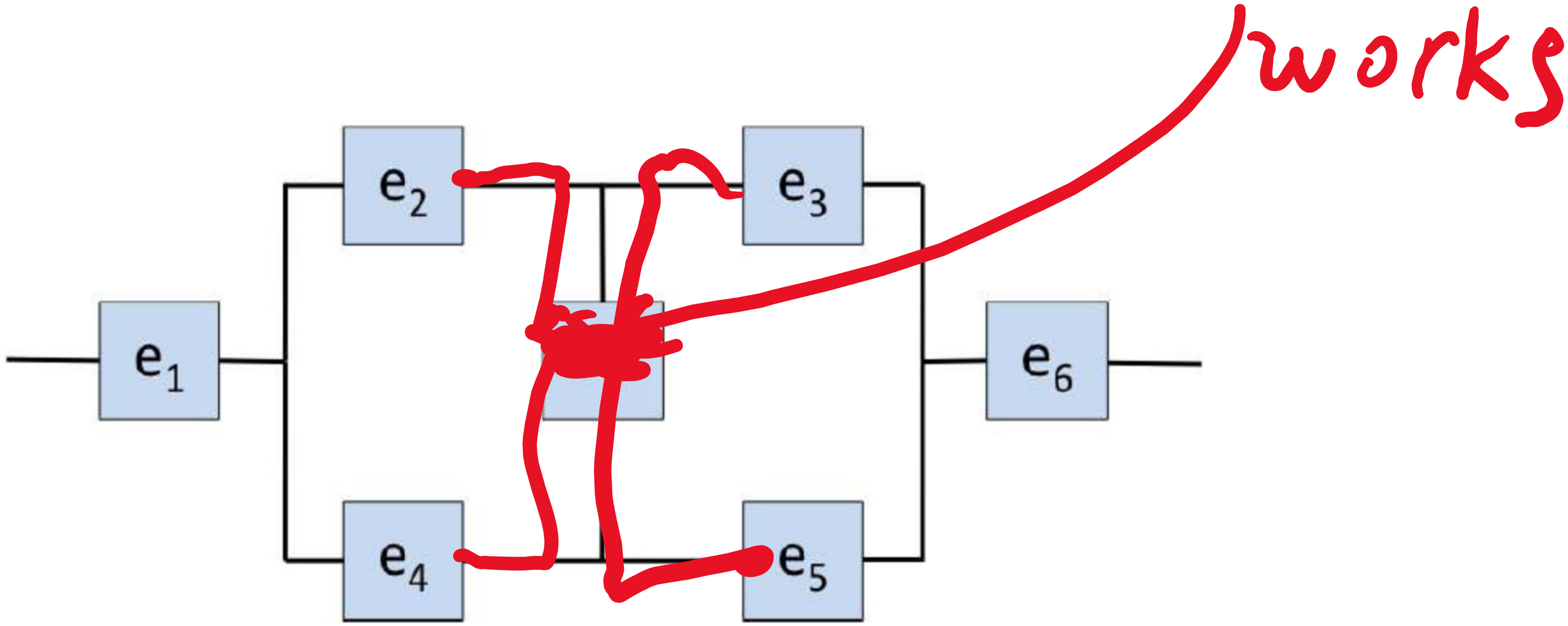

| Component | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| Probability of component working | 0.3 | 0.8 | 0.2 | 0.2 | 0.5 | 0.6 | 0.4 |




| Component | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| Probability of component working | 0.3 | 0.8 | 0.2 | 0.2 | 0.5 | 0.6 | 0.4 |

P(circuit works |e7 is broken)=P(e1 works)*
[1-(1-P(e2 works)*P(e3 works))*(1-P(e4 works)*P(e5 works))]*
P(e6 works)=0.3*(1-(1-0.8*0.2)*(1-0.2*0.5))*0.6=0.0439

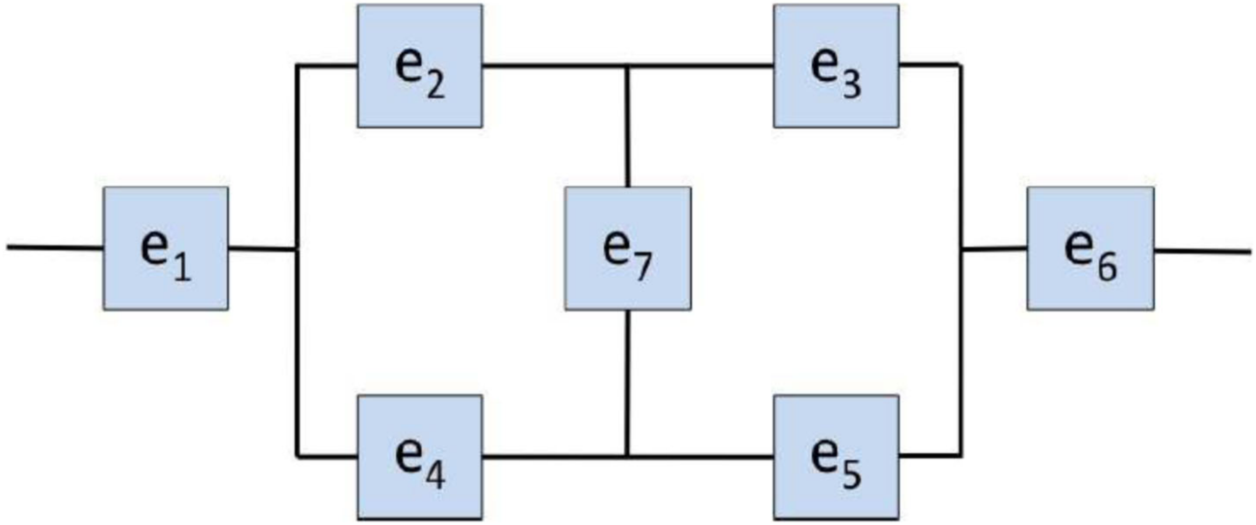
The contribution to total probability:
P(circuit works |e7 is broken)*P(e7 is broken)=0.6*0.0439=0.0264




| Component | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| Probability of component working | 0.3 | 0.8 | 0.2 | 0.2 | 0.5 | 0.6 | 0.4 |

P(circuit works |e7 works)=P(e1 works)*
[1-(1-P(e2 works))*(1-P(e3 works))]
*[1-(1-P(e4 works))*(1-P(e5 works))]*
P(e6 works)=0.3*(1-(1-0.8)*(1-0.2))*(1-(1-0.2)*(1-0.5)))*0.6=0.0907

The contribution to total probability:
P(circuit works |e7 works)*P(e7 works)=0.4*0.0907=0.0363

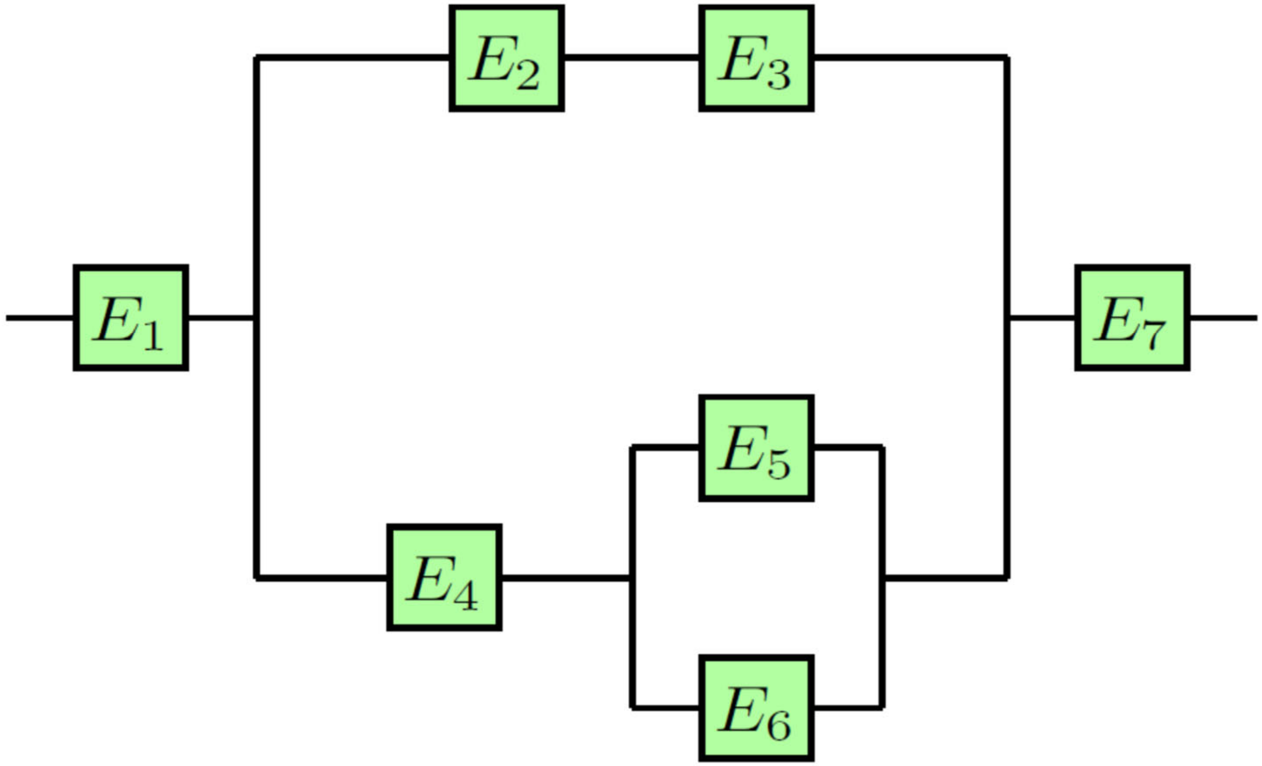



| Component | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| Probability of component working | 0.3 | 0.8 | 0.2 | 0.2 | 0.5 | 0.6 | 0.4 |

P(circuit works)=
P(circuit works |e7 works)*P(e7 works)+
P(circuit works |e7 is broken)*P(e7 is broken)=
=0.0264+0.0363=0.0627

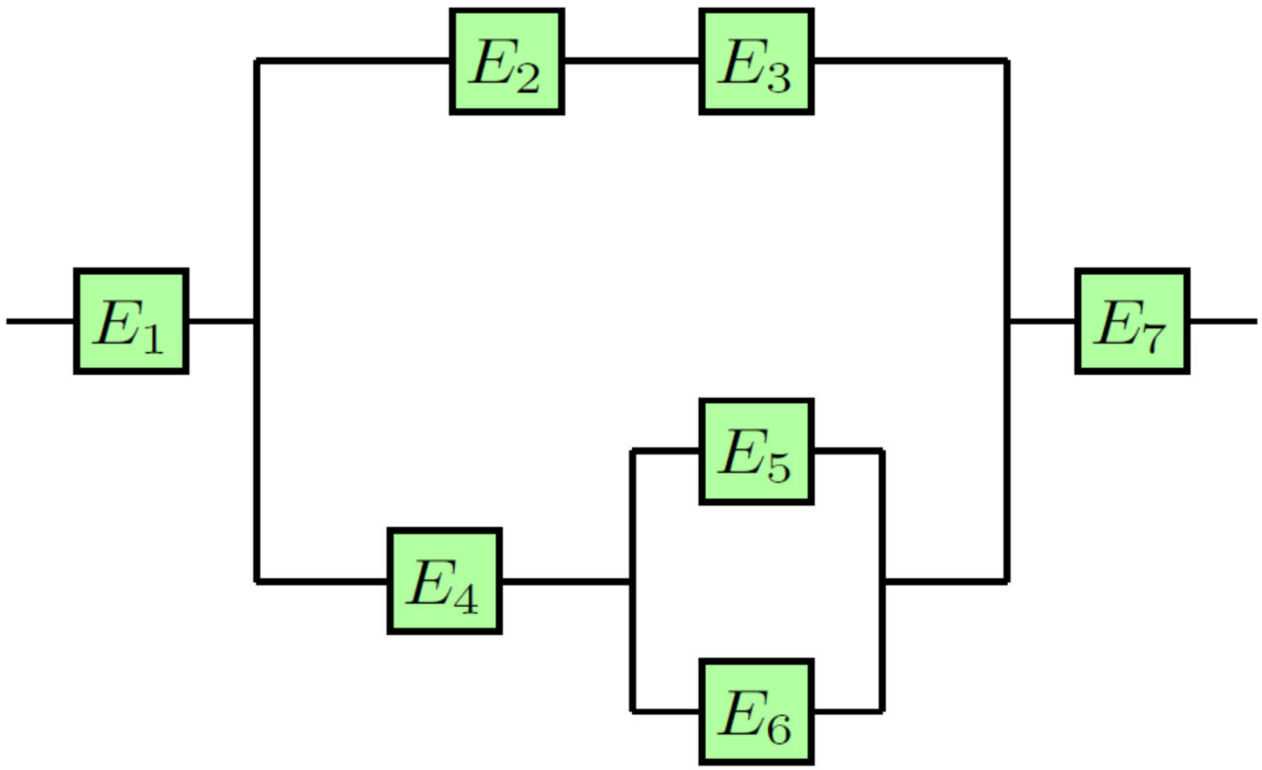
**Answer: 6.27%**

# Circuit → Set equation




| Component | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ |
|---|---|---|---|---|---|---|---|
| Probability of functioning well | 0.9 | 0.5 | 0.3 | 0.1 | 0.4 | 0.5 | 0.8 |

# Circuit → Set equation



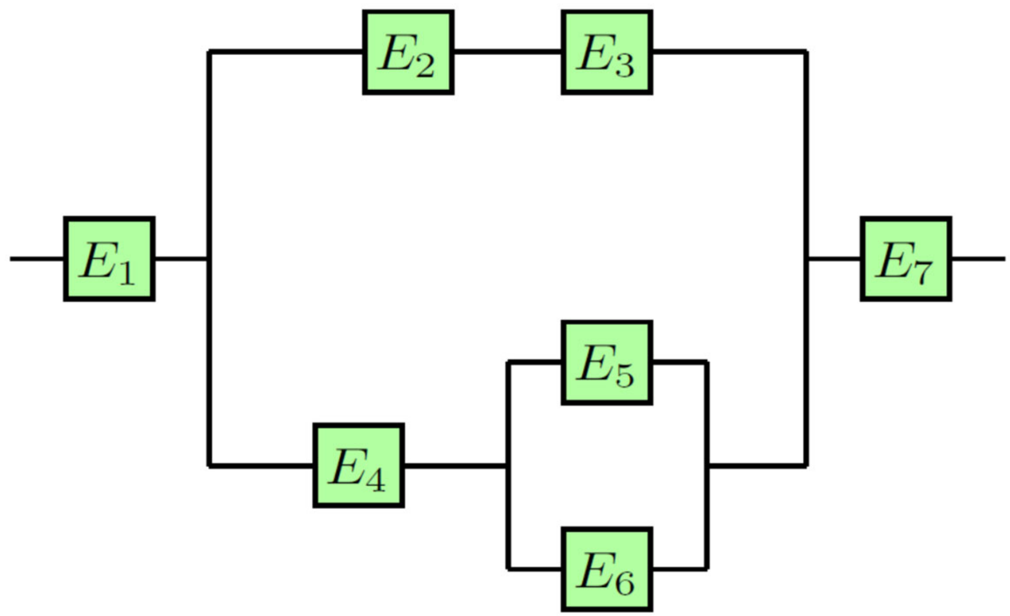

| Component | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ |
|---|---|---|---|---|---|---|---|
| Probability of functioning well | 0.9 | 0.5 | 0.3 | 0.1 | 0.4 | 0.5 | 0.8 |

$$E_1 \cap [(E_2 \cap E_3) \cup (E_4 \cap (E_5 \cup E_6))] \cap E_7.$$

P(Works) = 0.9.*(1-(1-0.5.*0.3).*(1-0.1.*(1-0.6.*0.5))).*0.8=0.15084

# Matlab group exercise

- Test our result for this circuit.
- Use circuit_template.m on the website




| Component | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ |
|---|---|---|---|---|---|---|---|
| Probability of functioning well | 0.9 | 0.5 | 0.3 | 0.1 | 0.4 | 0.5 | 0.8 |