

# Foundations of Probability

Random experiments

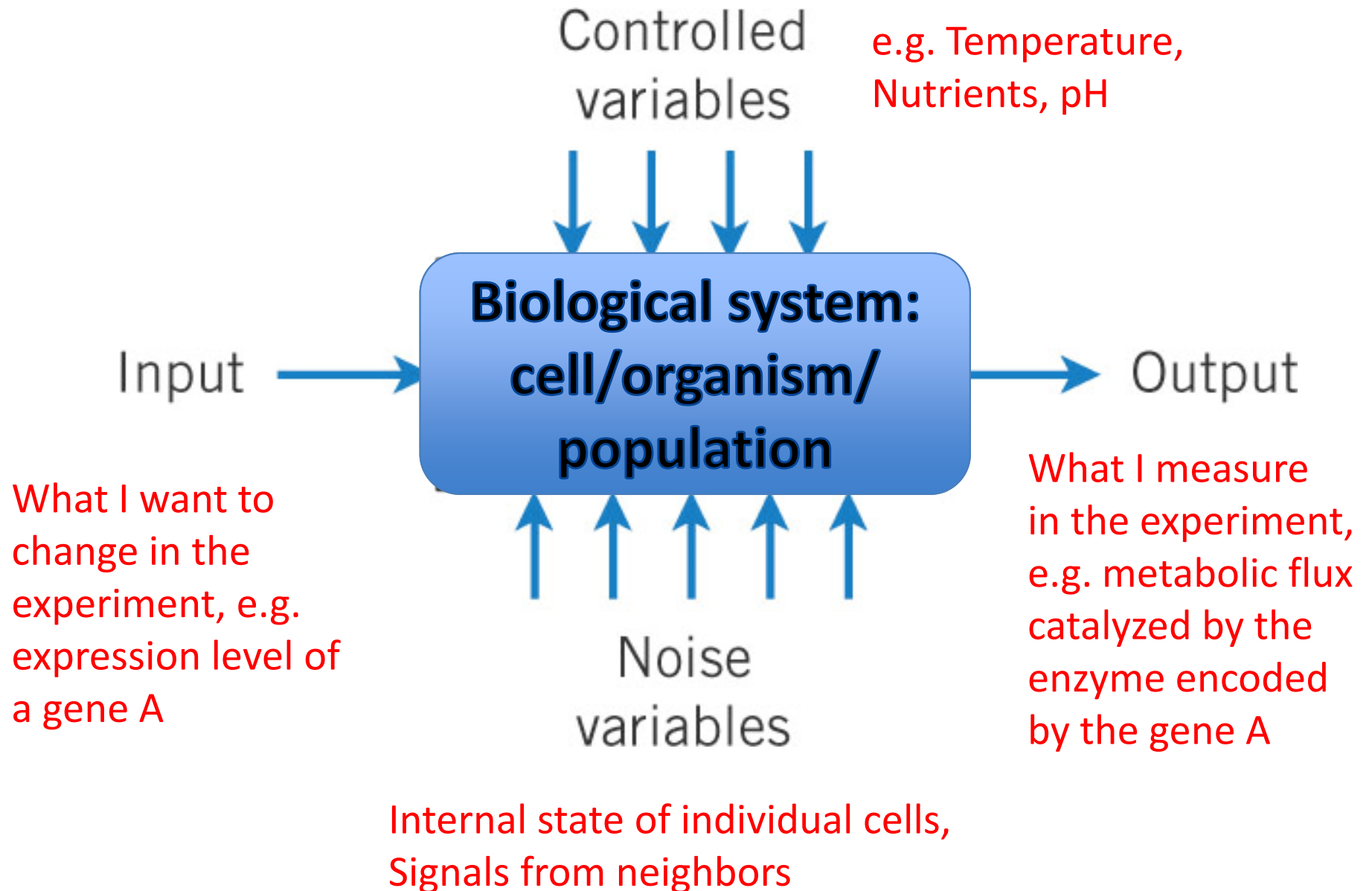
Sample spaces

Venn diagrams of  
random events

# Random Experiments

- An **experiment** is an operation or procedure, carried out under controlled conditions
  - Example: measure the metabolic flux through a reaction catalyzed by the enzyme A
- An experiment that can result in **different outcomes**, even if repeated in the same manner every time, is called a **random experiment**
  - Cell-to-cell variability due to history/genome variants
  - Noise in external parameters such as temperature, nutrients, pH, etc.
- **Evolution** offers ready-made random experiments
  - Genomes of different species
  - Genomes of different individuals within a species
  - Individual cancer cells

# Variability/Noise Produce Output Variation



# Sample Spaces

- **Random experiments** have **unique outcomes**.
- The set of **all possible outcomes** of a random experiment is called the **sample space,  $S$** .
- $S$  is **discrete** if it consists of a **finite** or **countable infinite** set of **outcomes**.
- $S$  is **continuous** if it contains an **interval** (either a finite or infinite width) **of real numbers**.

# Examples of a Sample Space

- Experiment measuring the abundance of mRNA expressed from a single gene

$S = \{x \mid x \geq 0\}$ : **continuous.**

- Bin it into four groups

$S = \{\textit{below 10, 10-30, 30-100, above 100}\}$ :  
**discrete.**

- Is gene “on” (mRNA above 30)?

$S = \{\textit{true, false}\}$ : **logical/Boolean/discrete.**

# Event

An event ( $E$ ) is a **subset of the sample space** of a random experiment, i.e., **one or more** outcomes of the sample space.

- The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as  $E_1 \cup E_2$
- The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as  $E_1 \cap E_2$
- The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the complement of the event  $E$  as  $E'$  (sometimes  $E^c$  or  $\bar{E}$  )

# Examples

## Discrete

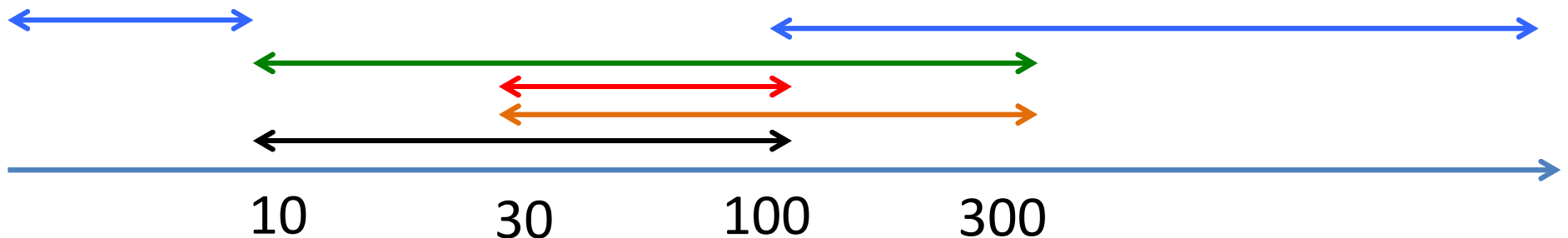
1. Assume you toss a coin once. The sample space is  $S = \{H, T\}$ , where H = head and T = tail and the event of a head is  $\{H\}$ .
2. Assume you toss a coin twice. The sample space is  $S = \{(H, H), (H, T), (T, H), (T, T)\}$ , and the event of obtaining exactly one head is  $\{(H, T), (T, H)\}$ .

## Continuous

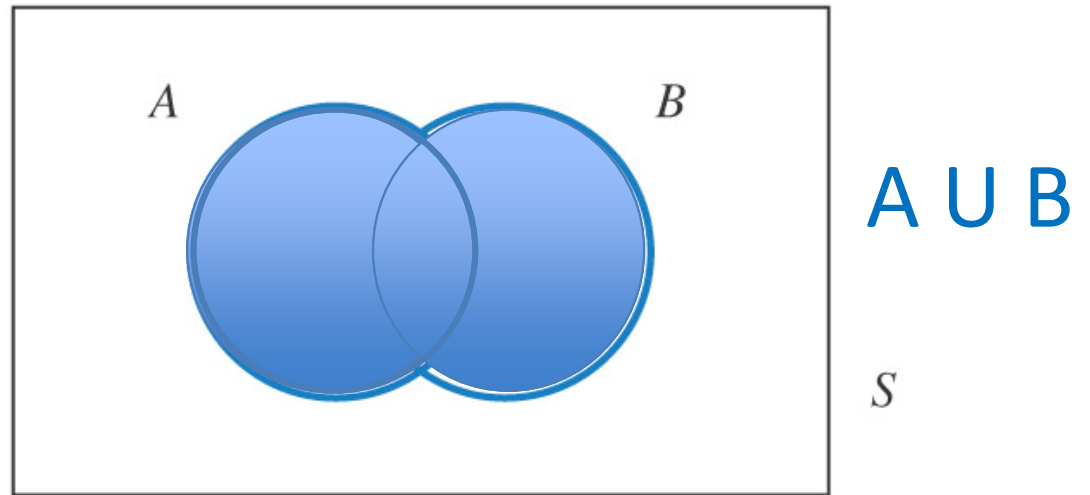
Sample space for the expression level of a gene:  $S = \{x | x \geq 0\}$

Two events:

- $E1 = \{x | 10 < x < 100\}$
- $E2 = \{x | 30 < x < 300\}$
- $E1 \cap E2 = \{x | 30 < x < 100\}$
- $E1 \cup E2 = \{x | 10 < x < 300\}$
- $E1' = \{x | x \leq 10 \text{ or } x \geq 100\}$



# Venn diagrams



John Venn (1843-1923)  
British logician

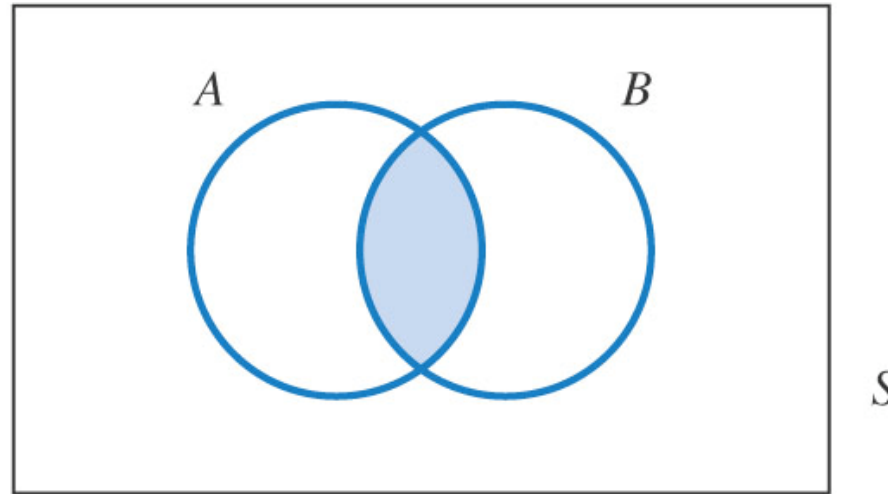
Find  
5 differences  
in beard and  
hairstyle



John Venn (1990- )  
Brooklyn hipster



# Venn diagrams

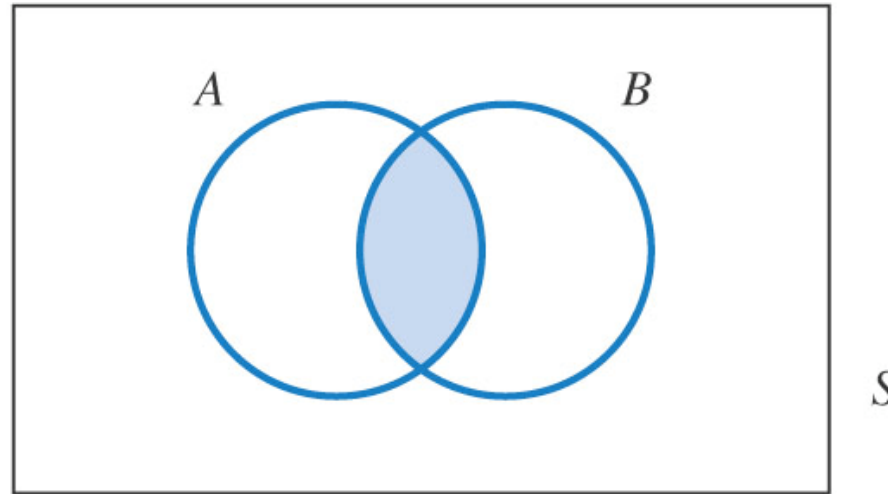


Which formula describes the blue region?

- A.  $A \cup B$
- B.  $A \cap B$
- C.  $A'$
- D.  $B'$

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# Venn diagrams



Which formula describes the blue region?

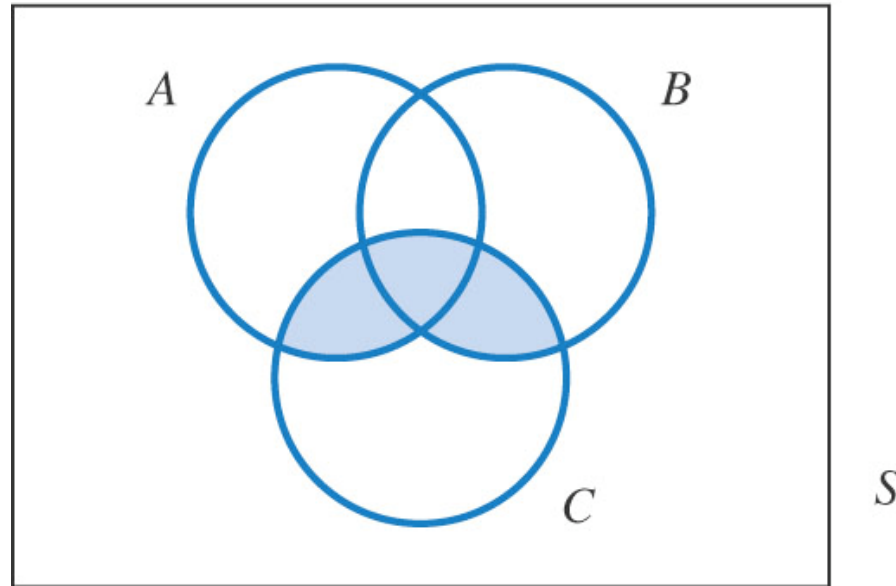
A.  $A \cup B$

B.  $A \cap B$

C.  $A'$

D.  $B'$

# Venn diagrams

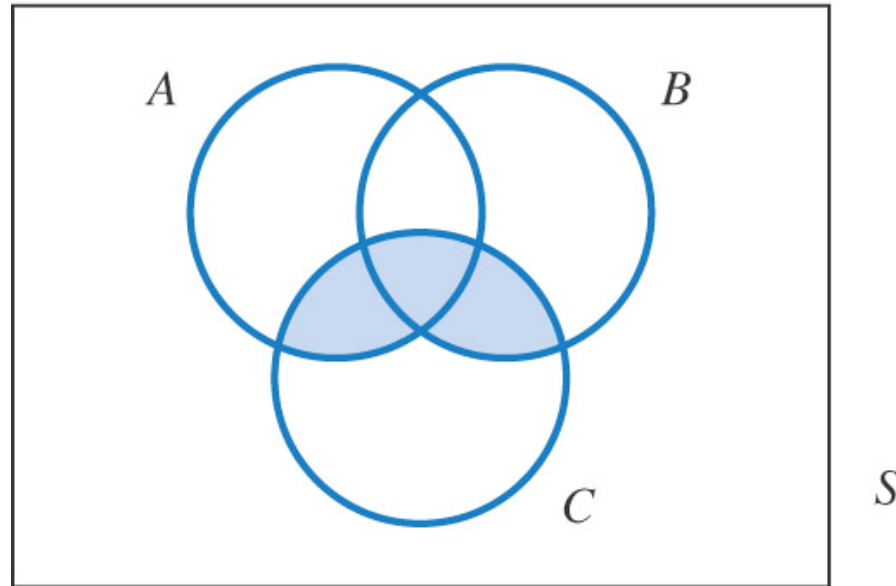


Which formula describes the blue region?

- A.  $(A \cup B) \cap C$
- B.  $(A \cap B) \cap C$
- C.  $(A \cup B) \cup C$
- D.  $(A \cap B) \cup C$

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# Venn diagrams



Which formula describes the blue region?

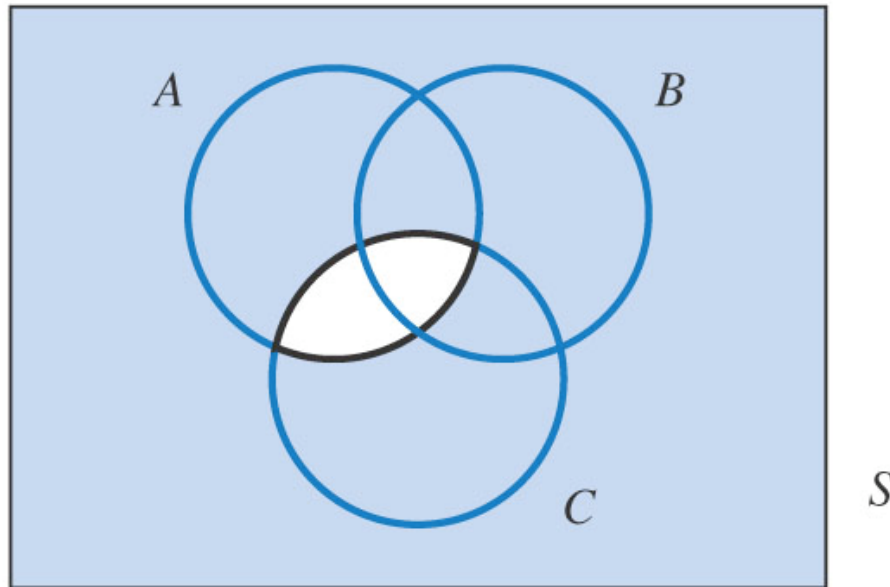
A.  $(A \cup B) \cap C$

B.  $(A \cap B) \cap C$

C.  $(A \cup B) \cup C$

D.  $(A \cap B) \cup C$

# Venn diagrams

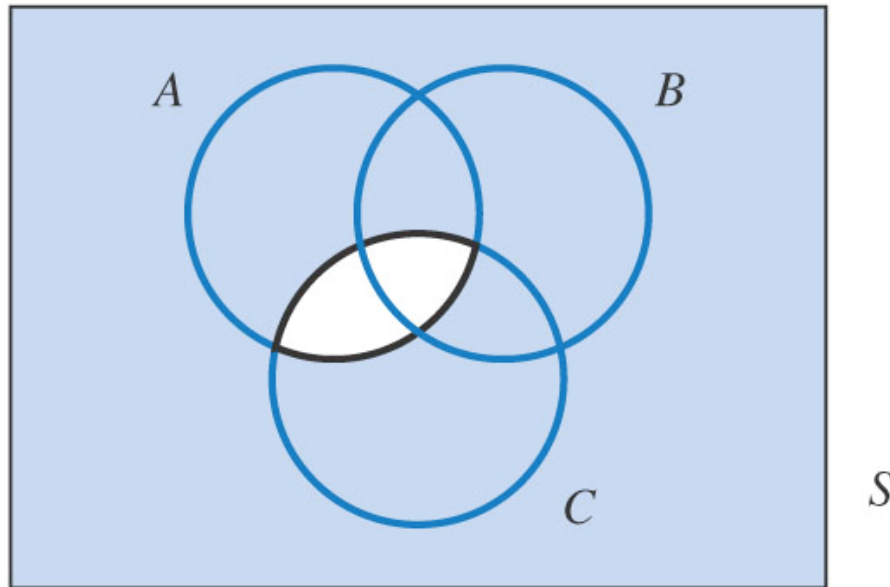


Which formula describes the blue region?

- A.  $A \cap C$
- B.  $A' \cup C'$
- C.  $(A \cap B \cap C)'$
- D.  $(A \cap B) \cap C$

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# Venn diagrams



Which formula describes the blue region?

A.  $A \cap C$

B.  $A' \cup C'$

C.  $(A \cap B \cap C)'$

D.  $(A \cap B) \cap C$

Credit: XKCD  
comics

# WHY ARE THERE SLAVES IN THE BIBLE

WHY DO TWINS HAVE DIFFERENT FINGERPRINTS  
WHY ARE AMERICANS AFRAID OF DRAGONS

WHY IS HTTPS CROSSED OUT IN RED  
WHY IS THERE A LINE THROUGH HTTPS  
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK  
WHY IS HTTPS IMPORTANT

# QUESTIONS FOUND IN GOOGLE AUTOCOMPLETE



WHY ARE THERE WEEKS  
WHY DO I FEEL DIZZY

WHY DO WHALES JUMP  
WHY ARE WITCHES GREEN  
WHY ARE THERE MIRRORS ABOVE BEDS  
WHY DO I SAY UH  
WHY IS SEA SALT BETTER  
WHY ARE THERE TREES IN THE MIDDLE OF FIELDS  
WHY IS THERE NOT A POKEMON MMO  
WHY IS THERE LAUGHING IN TV SHOWS  
WHY ARE THERE DOORS ON THE FREEWAY  
WHY ARE THERE SO MANY SVCHOST.EXE RUNNING  
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA  
WHY ARE THERE SCARY SOUNDS IN MINECRAFT  
WHY IS THERE KICKING IN MY STOMACH  
WHY ARE THERE TWO SLASHES AFTER HTTP  
WHY ARE THERE CELEBRITIES  
WHY DO SNAKES EXIST  
WHY DO OYSTERS HAVE PEARLS  
WHY ARE DUCKS CALLED DUCKS  
WHY DO THEY CALL IT THE CLAP  
WHY ARE KYLE AND CARTMAN FRIENDS  
WHY IS THERE AN ARROW ON AANG'S HEAD  
WHY ARE TEXT MESSAGES BLUE  
WHY ARE THERE MUSTACHES ON CLOTHES  
WHY ARE THERE MUSTACHES ON CARS  
WHY ARE THERE MUSTACHES EVERYWHERE  
WHY ARE THERE SO MANY BIRDS IN OHIO  
WHY IS THERE SO MUCH RAIN IN OHIO  
WHY IS OHIO WEATHER SO WEIRD

WHY AREN'T ECONOMISTS RICH  
WHY DO AMERICANS CALL IT SOCCER  
WHY ARE MY EARS RINGING  
WHY ARE THERE SO MANY AVENGERS  
WHY ARE THE AVENGERS FIGHTING THE X MEN  
WHY IS WOLVERINE NOT IN THE AVENGERS

WHY ARE THERE SWARMS OF GNATS  
WHY IS THERE PHLEGM  
WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN  
WHY IS PSYCHIC WEAK TO BUG  
WHY DO CHILDREN GET CANCER  
WHY IS POSEIDON ANGRY WITH ODYSSEUS  
WHY IS THERE ICE IN SPACE

# WHY ARE THERE ANTS IN MY LAPTOP

WHY ARE THERE BRIDESMAIDS  
WHY DO DYING PEOPLE REACH UP  
WHY AREN'T THERE VARICOSE ARTERIES  
WHY ARE OLD KUNGONS DIFFERENT



WHY IS EARTH TILTED  
WHY IS SPACE BLACK  
WHY IS OUTER SPACE SO COLD  
WHY ARE THERE PYRAMIDS ON THE MOON  
WHY IS NASA SHUTTING DOWN



WHY IS THERE AN OWL IN MY BACKYARD  
WHY IS THERE AN OWL OUTSIDE MY WINDOW  
WHY IS THERE AN OWL ON THE DOLLAR BILL  
WHY DO OWLS ATTACK PEOPLE  
WHY ARE AK 47s SO EXPENSIVE  
WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE  
WHY ARE THERE GODS  
WHY ARE THERE TWO SPOCKS

WHY ARE DOGS AFRAID OF FIREWORKS  
WHY IS THERE NO KING IN ENGLAND

WHY IS PROGRAMMING SO HARD  
WHY IS THERE A 0 OHM RESISTOR  
WHY DO AMERICANS HATE SOCCER  
WHY DO RHYMES SOUND GOOD  
WHY DO TREES DIE  
WHY IS THERE NO SOUND ON CNN  
WHY AREN'T POKEMON REAL  
WHY AREN'T BULLETS SHARP  
WHY DO DREAMS SEEM SO REAL

WHY ARE THERE TINY SPIDERS IN MY HOUSE  
WHY DO SPIDERS COME INSIDE  
WHY ARE THERE HUGE SPIDERS IN MY HOUSE  
WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE  
WHY ARE THERE SPIDERS IN MY ROOM  
WHY ARE THERE SO MANY SPIDERS IN MY ROOM  
WHY DO SPIDER BITES ITCH  
WHY IS DYING SO SCARY

WHY IS THERE NO GPS IN LAPTOPS  
WHY DO KNEES CLICK  
WHY AREN'T THERE E GRADES  
WHY IS ISOLATION BAD  
WHY DO BOYS LIKE ME  
WHY DON'T BOYS LIKE ME  
WHY IS THERE ALWAYS A JAVA UPDATE  
WHY ARE THERE RED DOTS ON MY THIGHS  
WHY IS LYING GOOD



WHY IS MT VESUVIUS THERE  
WHY DO THEY SAY T MINUS  
WHY ARE THERE OBELISKS  
WHY ARE WRESTLERS ALWAYS WET  
WHY ARE OCEANS BECOMING MORE ACIDIC  
WHY IS ARWEN DYING  
WHY AREN'T MY QUAIL LAYING EGGS  
WHY AREN'T MY QUAIL EGGS HATCHING  
WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA

WHY ARE CIGARETTES LEGAL  
WHY ARE THERE DUCKS IN MY POOL  
WHY IS JESUS WHITE  
WHY IS THERE LIQUID IN MY EAR  
WHY DO Q TIPS FEEL GOOD  
WHY DO GOOD PEOPLE DIE



WHY ARE ULTRASOUNDS IMPORTANT  
WHY ARE ULTRASOUND MACHINES EXPENSIVE  
WHY IS STEALING WRONG

# Definitions of Probability



# Two definitions of probability

- (1) **STATISTICAL PROBABILITY**: the relative frequency with which an event occurs in the long run
- (2) **INDUCTIVE PROBABILITY**: the degree of belief which it is reasonable to place in a proposition on given evidence

# Statistical Probability

A **statistical probability** of an event is the **limiting value** of the **relative frequency** with it occurs in a **very large number** of **independent trials**

Empirical

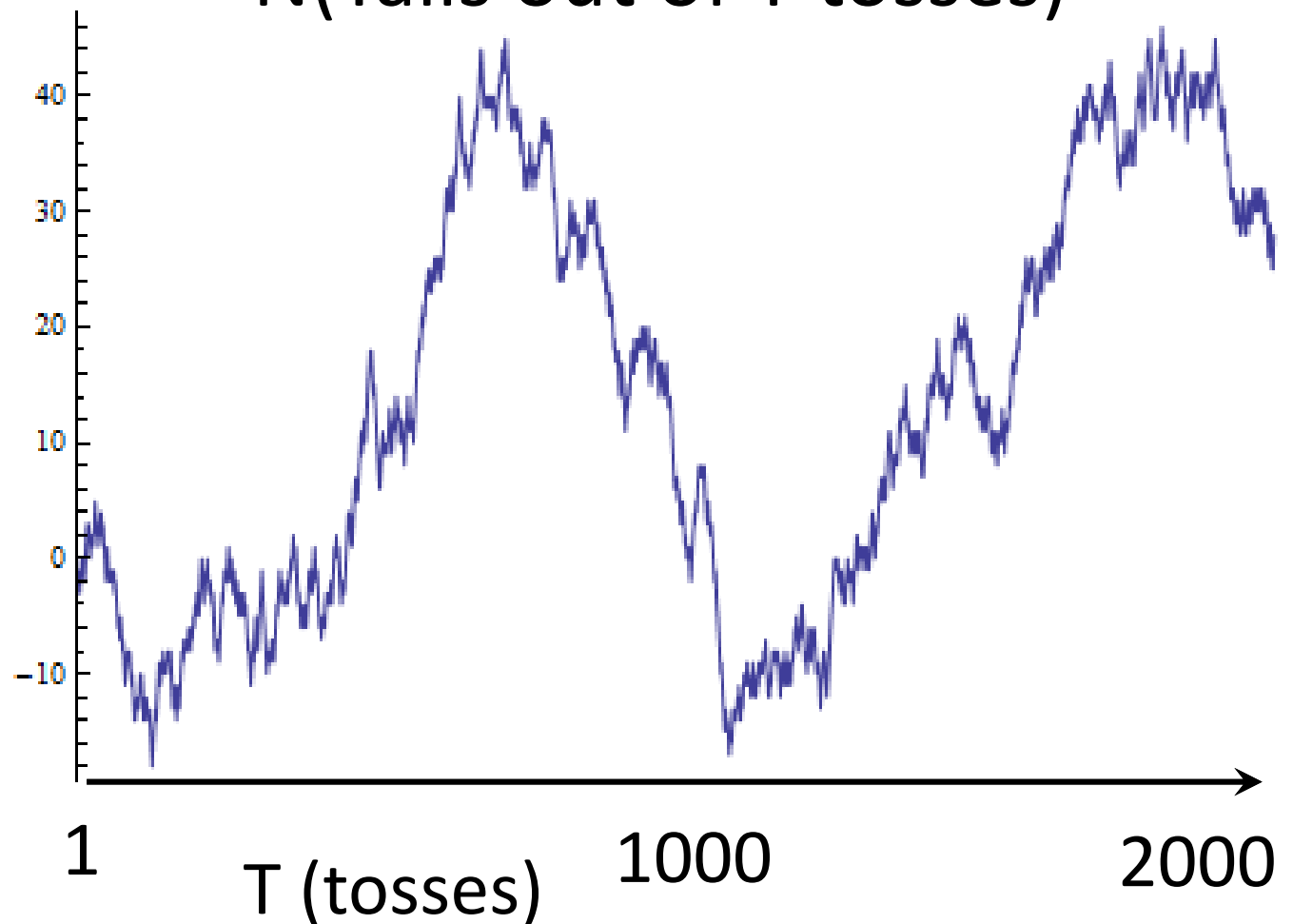
# Statistical Probability of a Coin Toss



**John Edmund Kerrich**  
(1903–1985)  
British/South African  
mathematician

$N(\text{Heads out of } T \text{ tosses}) -$   
 $-N(\text{Tails out of } T \text{ tosses})$

Net 1 vs. 0

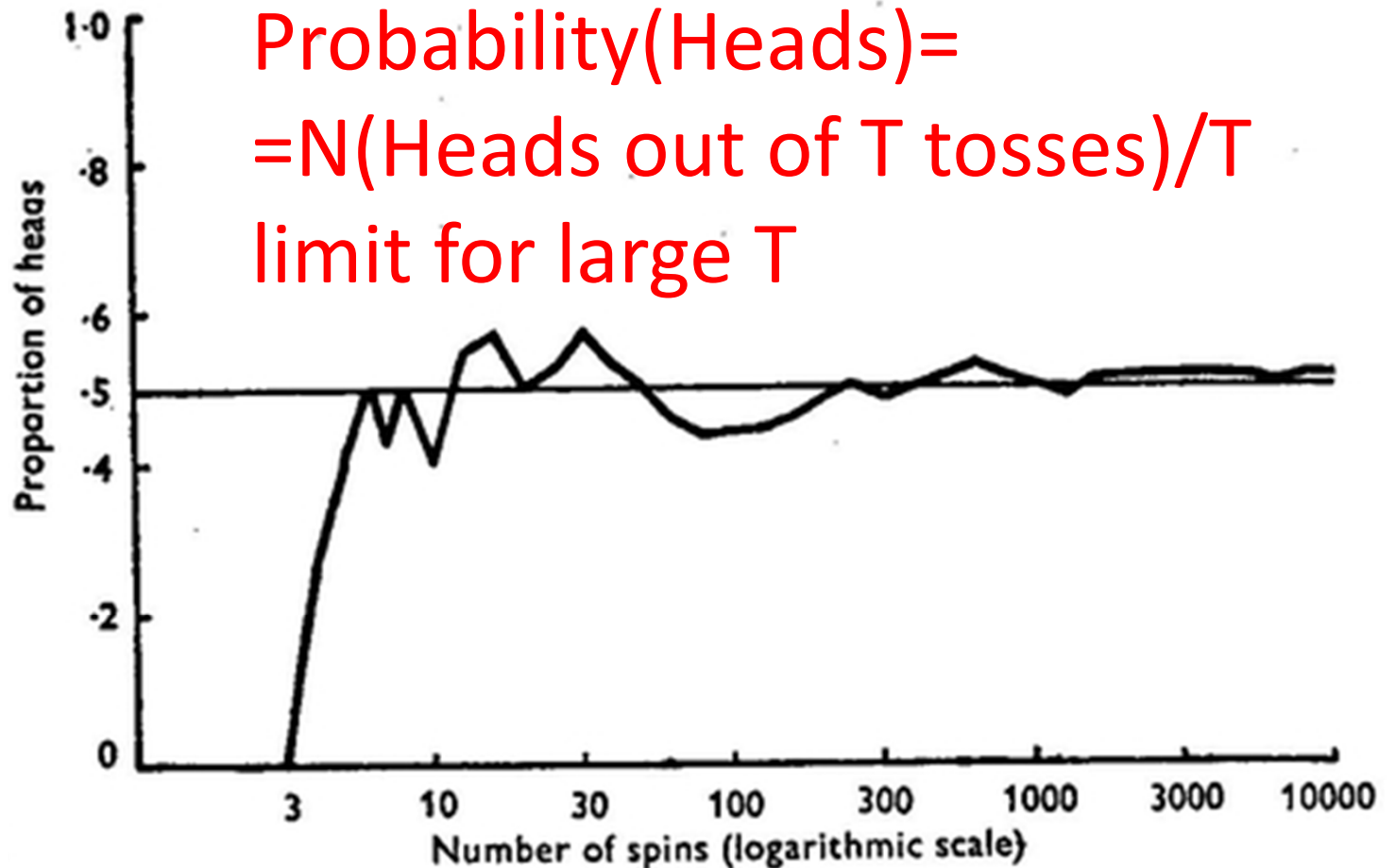


Excess of heads among 2,000 coin tosses (Kerrich 1946)

# Statistical Probability of a Coin Toss



**John Edmund Kerrich**  
(1903–1985)  
British/South African  
mathematician



Proportion of heads among 10,000 coin tosses (Kerrich 1946)

# Who is ready to use Matlab?

- A. I have Matlab installed on my laptop
- B. I am ready to use Matlab on EWS
- C. I don't have it ready but plan to install it
- D. I am not ready but plan to use EWS
- E. I plan to use other software (Python, R, etc.)

Get your i-clickers

# Matlab is easy to learn

- Matlab is the lingua franca of all of **engineering**
- Use online tutorials e.g.:  
<https://www.youtube.com/watch?v=82TGgQApFIQ>
- **Matlab** is designed to work with **Matrices** → symbols **\*** and **/** are understood as **matrix multiplication** and **division**
- Use **.\*** and **./** for regular (non-matrix) multiplication
- Add **;** in the end of the line to avoid displaying the output on the screen
- **Loops**: `for i=1:100; f(i)=floor(2.*rand); end;`
- **Conditional statements**: `if rand>0.5; count=count+1; end;`
- **Plotting**: `plot(x,y,'ko-')`; or `semilogx(x,y,'ko-')`; or `loglog(x,y,'ko-')`; .  
To keep **adding plots onto the same axes** use: `hold on;`  
To **create a new axes** use `figure;`
- **Generating matrices**: `rand(100)` – generates square matrix 100x100.  
**Confusing!** Use `rand(100,1)` or `zeros(30,20)`, or `randn(1,40)` (Gaussian);
- If Matlab complains multiplying matrices **check sizes** using `whos` and if needed **use transpose** operation: `x=x'`;

# A Matlab Cheat-sheet (MIT 18.06, Fall 2007)

## Basics:

save 'file.mat' save variables to *file.mat*  
load 'file.mat' load variables from *file.mat*  
diary on record input/output to file *diary*  
diary off stop recording  
whos list all variables currently defined  
clear delete/undefine all variables  
help command quick help on a given *command*  
doc command extensive help on a given *command*

## Defining/changing variables:

$x = 3$  define variable  $x$  to be 3  
 $x = [1 \ 2 \ 3]$  set  $x$  to the  $1 \times 3$  row-vector (1,2,3)  
 $x = [1 \ 2 \ 3];$  same, but don't echo  $x$  to output  
 $x = [1;2;3]$  set  $x$  to the  $3 \times 1$  column-vector (1,2,3)  
 $A = [1 \ 2 \ 3 \ 4; 5 \ 6 \ 7 \ 8; 9 \ 10 \ 11 \ 12];$   
set  $A$  to the  $3 \times 4$  matrix with rows 1,2,3,4 etc.  
 $x(2) = 7$  change  $x$  from (1,2,3) to (1,7,3)  
 $A(2,1) = 0$  change  $A_{2,1}$  from 5 to 0

## Arithmetic and functions of numbers:

$3*4$ ,  $7+4$ ,  $2-6$   $8/3$  multiply, add, subtract, and divide numbers  
 $3^7$ ,  $3^{(8+2i)}$  compute 3 to the 7th power, or 3 to the  $8+2i$  power  
 $\text{sqrt}(-5)$  compute the square root of  $-5$   
 $\text{exp}(12)$  compute  $e^{12}$   
 $\text{log}(3)$ ,  $\text{log}_{10}(100)$  compute the natural log (ln) and base-10 log ( $\text{log}_{10}$ )  
 $\text{abs}(-5)$  compute the absolute value  $|-5|$   
 $\text{sin}(5*\text{pi}/3)$  compute the sine of  $5\pi/3$   
 $\text{besselj}(2,6)$  compute the Bessel function  $J_2(6)$

## Arithmetic and functions of vectors and matrices:

$x * 3$  multiply every element of  $x$  by 3  
 $x + 2$  add 2 to every element of  $x$   
 $x + y$  element-wise addition of two vectors  $x$  and  $y$   
 $A * y$  product of a matrix  $A$  and a vector  $y$   
 $A * B$  product of two matrices  $A$  and  $B$   
 $x * y$  not allowed if  $x$  and  $y$  are two column vectors!  
 $x .* y$  element-wise product of vectors  $x$  and  $y$   
 $A^3$  the square matrix  $A$  to the 3rd power  
 $x^3$  not allowed if  $x$  is not a square matrix!  
 $x.^3$  every element of  $x$  is taken to the 3rd power  
 $\text{cos}(x)$  the cosine of every element of  $x$   
 $\text{abs}(A)$  the absolute value of every element of  $A$   
 $\text{exp}(A)$   $e$  to the power of every element of  $A$   
 $\text{sqrt}(A)$  the square root of every element of  $A$   
 $\text{expm}(A)$  the matrix exponential  $e^A$   
 $\text{sqrtm}(A)$  the matrix whose square is  $A$

## Transposes and dot products:

$x.'$ ,  $A.'$  the transposes of  $x$  and  $A$   
 $x'$ ,  $A'$  the complex-conjugate of the transposes of  $x$  and  $A$   
 $x' * y$  the dot (inner) product of two column vectors  $x$  and  $y$   
 $\text{dot}(x, y)$ ,  $\text{sum}(x.*y)$  ...two other ways to write the dot product  
 $x * y'$  the outer product of two column vectors  $x$  and  $y$

## Constructing a few simple matrices:

$\text{rand}(12,4)$  a  $12 \times 4$  matrix with uniform random numbers in  $[0,1)$   
 $\text{randn}(12,4)$  a  $12 \times 4$  matrix with Gaussian random (center 0, variance 1)  
 $\text{zeros}(12,4)$  a  $12 \times 4$  matrix of zeros  
 $\text{ones}(12,4)$  a  $12 \times 4$  matrix of ones  
 $\text{eye}(5)$  a  $5 \times 5$  identity matrix  $I$  ("eye")  
 $\text{eye}(12,4)$  a  $12 \times 4$  matrix whose first 4 rows are the  $4 \times 4$  identity  
 $\text{linspace}(1.2, 4.7, 100)$   
row vector of 100 equally-spaced numbers from 1.2 to 4.7  
 $7:15$  row vector of 7,8,9,...,14,15  
 $\text{diag}(x)$  matrix whose diagonal is the entries of  $x$  (and other elements = 0)

## Portions of matrices and vectors:

$x(2:12)$  the 2nd to the 12th elements of  $x$   
 $x(2:\text{end})$  the 2nd to the last elements of  $x$   
 $x(1:3:\text{end})$  every third element of  $x$ , from 1st to the last  
 $x(:)$  all the elements of  $x$   
 $A(5,:)$  the row vector of every element in the 5th row of  $A$   
 $A(5,1:3)$  the row vector of the first 3 elements in the 5th row of  $A$   
 $A(:,2)$  the column vector of every element in the 2nd column of  $A$   
 $\text{diag}(A)$  column vector of the diagonal elements of  $A$

## Solving linear equations:

$A \setminus b$  for  $A$  a matrix and  $b$  a column vector, the solution  $x$  to  $Ax=b$   
 $\text{inv}(A)$  the inverse matrix  $A^{-1}$   
 $[L, U, P] = \text{lu}(A)$  the LU factorization  $PA=LU$   
 $\text{eig}(A)$  the eigenvalues of  $A$   
 $[V, D] = \text{eig}(A)$  the columns of  $V$  are the eigenvectors of  $A$ , and the diagonals  $\text{diag}(D)$  are the eigenvalues of  $A$

## Plotting:

$\text{plot}(y)$  plot  $y$  as the  $y$  axis, with 1,2,3,... as the  $x$  axis  
 $\text{plot}(x, y)$  plot  $y$  versus  $x$  (must have same length)  
 $\text{plot}(x, A)$  plot columns of  $A$  versus  $x$  (must have same # rows)  
 $\text{loglog}(x, y)$  plot  $y$  versus  $x$  on a log-log scale  
 $\text{semilogx}(x, y)$  plot  $y$  versus  $x$  with  $x$  on a log scale  
 $\text{semilogy}(x, y)$  plot  $y$  versus  $x$  with  $y$  on a log scale  
 $\text{fplot}(@(\text{x}) \dots \text{expression} \dots, [a, b])$   
plot some expression in  $x$  from  $x=a$  to  $x=b$   
 $\text{axis equal}$  force the  $x$  and  $y$  axes of the current plot to be scaled equally  
 $\text{title}('A Title')$  add a title  $A Title$  at the top of the plot  
 $\text{xlabel}('blah')$  label the  $x$  axis as *blah*  
 $\text{ylabel}('blah')$  label the  $y$  axis as *blah*  
 $\text{legend}('foo', 'bar')$  label 2 curves in the plot *foo* and *bar*  
 $\text{grid}$  include a grid in the plot  
 $\text{figure}$  open up a new figure window





# VIA app by Kramer needs to be updated

- Get the latest version app from <https://k.kramerav.com/support/download.asp?f=61213>
- On 8/24/2023 the version that worked was 4.0.3.1344

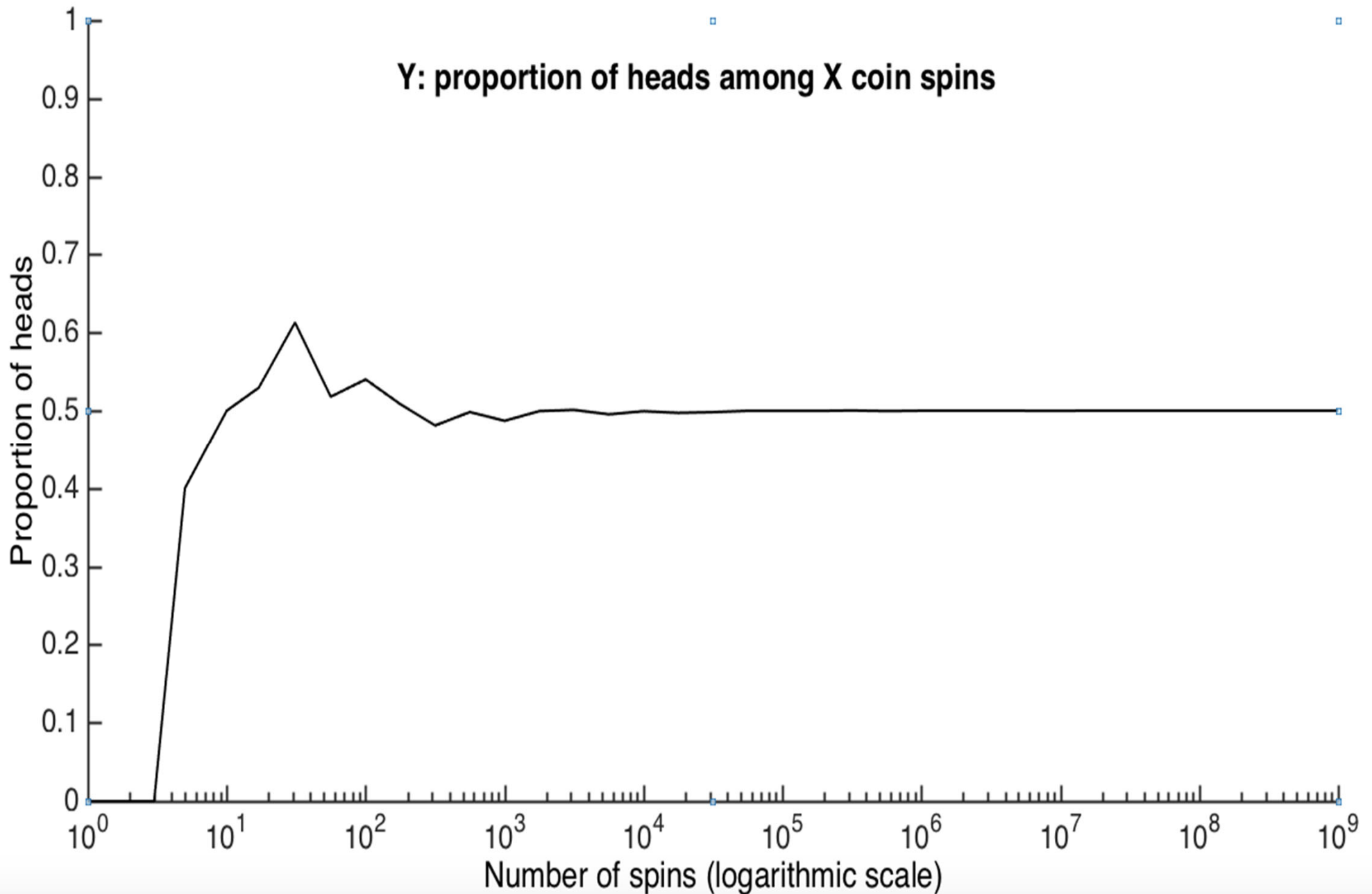
# Matlab group exercise

Each table to edit the file `coin_toss_template.m` (replace all ?? with commands/variables/operations ) or writes a new Matlab (Python, R, or anything else) script to:

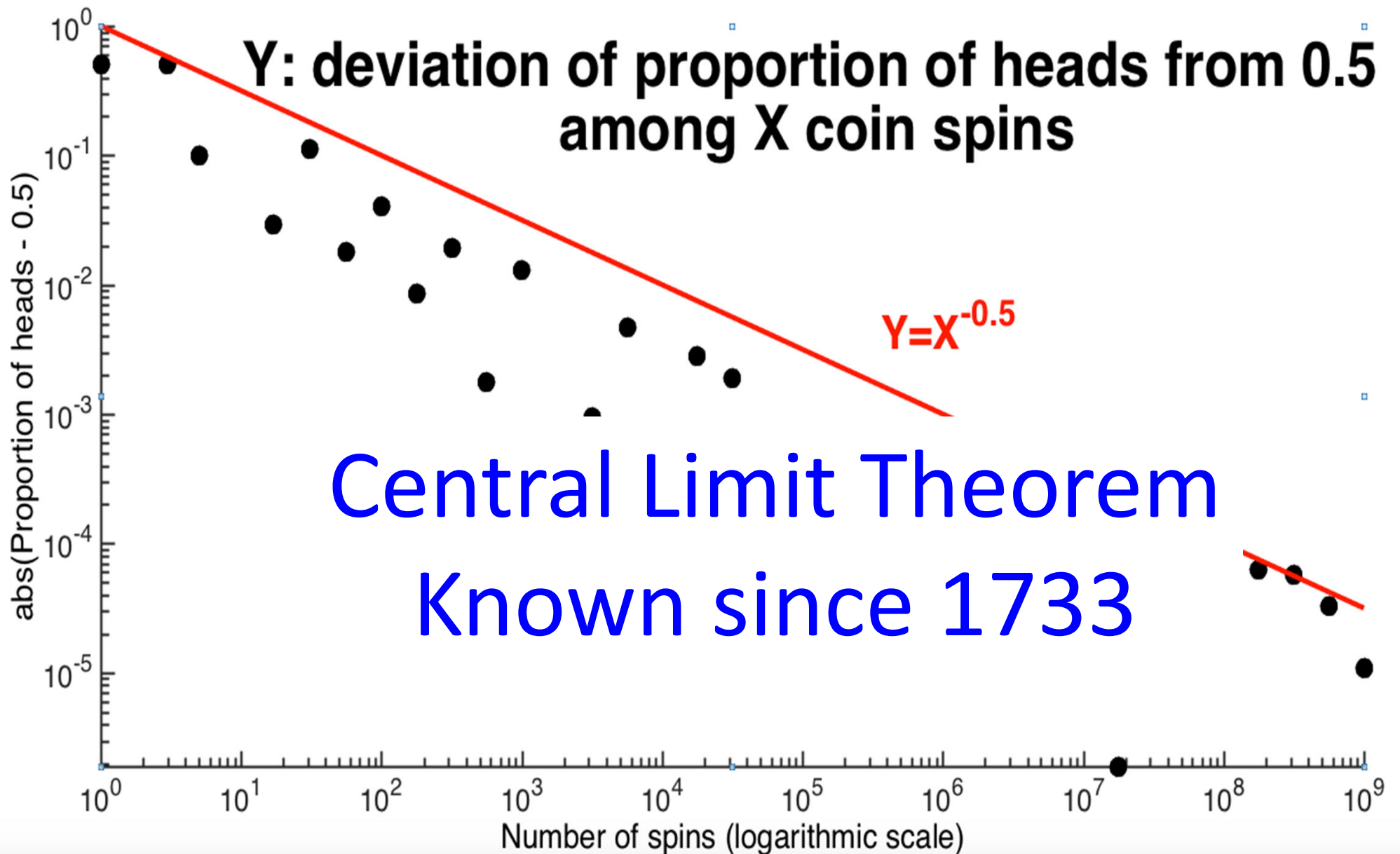
- Simulate a fair coin toss experiment
- Generate multiple tosses of a fair coin:  
1 – heads, 0 - tails
- Calculate the fraction of heads ( $f\_heads(t)$ ) at timepoints:  
t=10; 100; 1000; 10,000; 100,000; 1,000,000;10,000,000  
coin tosses
- Plot fraction of heads  $f\_heads(t)$  vs t with a **logarithmic t-axis**
- Plot  $abs(f\_heads(t)-0.5)$  vs t on a log-log plot (both axes are logarithmic)

# How I did it

- `Stats=1e7;`
- `r0=rand(Stats,1); r1=floor(2.*r0);`
- `n_heads(1)=r1(1);`
- `for t=2:Stats; n_heads(t)=n_heads(t-1)+r1(t); end;`
- `tp=[1, 10,100,1000, 10000, 100000, 1000000, 10000000]`
- `np=n_heads(tp); fp=np./tp`
- `figure; semilogx(tp,fp,'ko-');`
- `hold on; semilogx([1,10000000],[0.5,0.5], 'r--');`
- `figure; loglog(tp,abs(fp-0.5),'ko-');`
- `hold on; loglog(tp,0.5./sqrt(tp),'r--');`



Proportion of heads among 1,000,000,000 coin tosses  
( $10^5$  more than Kerrich) took me 33 seconds on my Surface Book



ABS(Proportion of heads-0.5)  
among 100,000,000 coin tosses