

I only received project writeup
from groups 1, 2 and 4

Group 3: please, send it today

Descriptive statistics:

Point estimation:

Point Estimation

- A sample was collected: X_1, X_2, \dots, X_n
- We suspect that sample was drawn from a random variable distribution $f(x)$
- $f(x)$ has k parameters that we do not know
- Point estimates are estimates of the parameters of the $f(x)$ describing the population based on the sample
 - For exponential PDF: $f(x) = \lambda \exp(-\lambda x)$ one wants to estimate λ
 - For Bernoulli PDF: $p^x(1-p)^{1-x}$ one wants to estimate p
 - For normal PDF one wants to estimate both μ and σ
- Point estimates are uncertain: therefore, we can talk of averages and standard deviations of point estimators

Point Estimator

A **point estimate** of some parameter θ describing population random variable is a single numerical value $\hat{\theta}$ depending on all values x_1, x_2, \dots, x_n in the sample.

The sample statistic (whis a random variable $\hat{\Theta}$ defined by a function $\hat{\Theta}(X_1, X_2, \dots, X_n)$) is called the **point estimator**.

- There could be **multiple choices** for the point estimator of a parameter.
- To estimate the **mean of a population**, we could choose the:
 - **Sample mean**
 - Sample median
 - Peak of the histogram
 - $\frac{1}{2}$ of (largest + smallest) observations of the sample.
- We need to develop criteria to compare estimates using statistical properties.

Unbiased Estimators Defined

The point estimator $\hat{\Theta}$ is an **unbiased estimator**

for the parameter θ if:

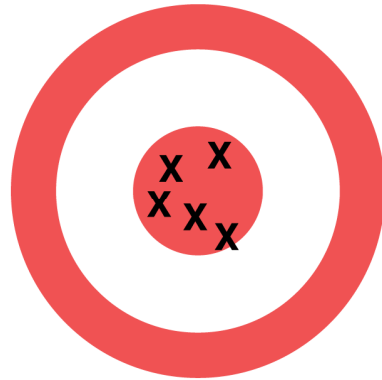
$$E(\hat{\Theta}) = \theta \quad (7-5)$$

If the estimator is not unbiased, then the difference:

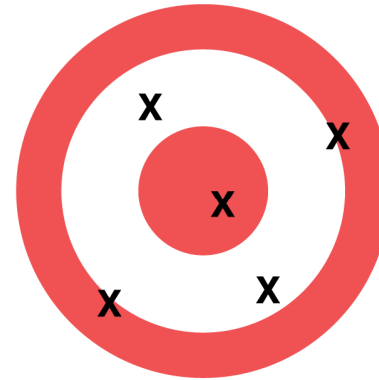
$$E(\hat{\Theta}) - \theta \quad (7-6)$$

is called the **bias** of the estimator $\hat{\Theta}$.

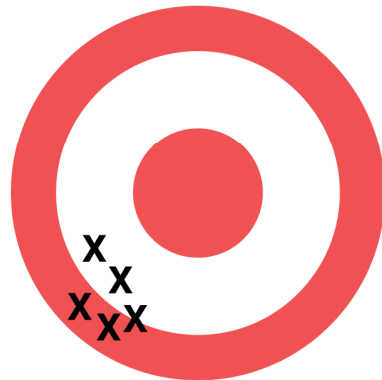
Bias vs Noise



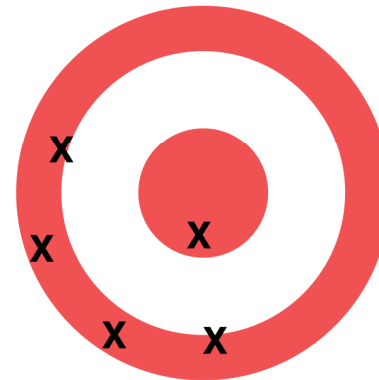
ACCURATE



NOISY



BIASED



BIASED & NOISY

Mean Squared Error

The **mean squared error** of an estimator $\hat{\Theta}$ of the parameter θ is defined as:

$$\text{MSE}(\hat{\Theta}) = E(\hat{\Theta} - \theta)^2 \quad (7-7)$$

Can be rewritten as

$$\begin{aligned} &= E[\hat{\Theta} - E(\hat{\Theta})]^2 + [\theta - E(\hat{\Theta})]^2 \\ &= V(\hat{\Theta}) + (\text{bias})^2 \end{aligned}$$

Statistic #1: Sample Mean

If the values of n observations in a random sample are denoted by x_1, x_2, \dots, x_n , the **sample mean** is

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum_{i=1}^n x_i}{n} \quad (6-1)$$

New random variable \bar{X} is a linear combination of n independent identically distributed variables X_1, X_2, \dots, X_n

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

Sample mean \bar{x} is drawn from a random variable

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$E(\bar{X}) = \frac{n \cdot E(X_i)}{n} = \frac{n \cdot \mu}{n} = \mu$$

Sample mean, \bar{X} , is an unbiased estimator of the population mean, μ

Sample variance S^2 –
is an estimator of
the population variance σ^2

Sample Variance

If n observations in a sample are denoted by x_1, x_2, \dots, x_n , the **sample variance** is

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1} \quad (6-3)$$

If one knows the **population average, μ** , one **divides by n** to estimate the variance

$$s(\mu)^2 = \frac{\sum_{i=1}^n (x_i - \mu)^2}{n}$$

Why divide by n-1 instead of n?

- The **sample mean \bar{x} is on average closer** to points x_1, x_2, \dots, x_n than **the true mean μ**
$$\sum_{i=1}^n (x_i - \bar{x})^2 \geq \sum_{i=1}^n (x_i - \mu)^2$$
- Consider a sample of size $n=1$.
Then $\bar{x} = x_1$ while $\mu \neq x_1$. Dividing by n gives $s^2 = 0$, while dividing by $n-1$ leaves **s^2 undefined (0/0)**
- For $n=2$, \bar{x} is exactly halfway between x_1 and x_2 making its **sum of squares smaller than** that of μ
- Dividing by $n-1$ on average corrects for a smaller sum of squares: **S^2 is an unbiased estimator of σ^2**

Show that s^2 is unbiased estimate of σ^2

$$\begin{aligned} E(s^2) &= E\left(\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}\right) = \frac{1}{n-1} E\left[\sum_{i=1}^n (X_i^2 + \bar{X}^2 - 2\bar{X}X_i)\right] \\ &= \frac{1}{n-1} E\left[\sum_{i=1}^n X_i^2 + n\bar{X}^2 - 2\bar{X}n\bar{X}\right] = \\ &= \frac{1}{n-1} E\left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right) = \frac{1}{n-1} (nE(X_i^2) - nE(\bar{X}^2)) \\ &= \frac{1}{n-1} \left(n\left(\mu^2 + \sigma^2\right) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right) = \frac{n-1}{n-1} \sigma^2 = \underline{\underline{\sigma^2}} \end{aligned}$$

Example 7-4: Sample Variance S^2 is Unbiased

$$\begin{aligned} E(S^2) &= E\left(\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}\right) \\ &= \frac{1}{n-1} E\left[\sum_{i=1}^n (X_i^2 + \bar{X}^2 - 2\bar{X}X_i)\right] \\ &= \frac{1}{n-1} \left[E\left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right) \right] \\ &= \frac{1}{n-1} \left[\sum_{i=1}^n (\mu^2 + \sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right) \right] \\ &= \frac{1}{n-1} [n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2] = \frac{1}{n-1} [(n-1)\sigma^2] \end{aligned}$$

Credit: XKCD
comics

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WHY ARE AMERICANS AFRAID OF DRAGONS

WHY IS HTTPS CROSSED OUT IN RED
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WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT

QUESTIONS

FOUND IN GOOGLE AUTOCOMPLETE



WHY ARE THERE WEEKS
WHY DO I FEEL DIZZY

WHY AREN'T ECONOMISTS RICH

WHY DO AMERICANS CALL IT SOCCER

WHY ARE MY EARS RINGING

WHY ARE THERE SO MANY AVENGERS

WHY ARE THE AVENGERS FIGHTING THE X MEN
WHY IS WOLVERINE NOT IN THE AVENGERS

WHY ARE THERE ANTS IN MY LAPTOP

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WHY IS THERE AN OWL ON THE DOLLAR BILL

WHY DO OWLS ATTACK PEOPLE

WHY ARE AK 47s SO EXPENSIVE

WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE

WHY ARE THERE GODS

WHY ARE THERE TWO SPOCKS

WHY IS MT VESUVIUS THERE

WHY DO THEY SAY T MINUS

WHY ARE THERE OBELISKS

WHY ARE WRESTLERS ALWAYS WET

WHY ARE OCEANS BECOMING MORE ACIDIC

WHY IS ARWEN DYING

WHY AREN'T MY QUAIL LAYING EGGS
WHY AREN'T MY QUAIL EGGS HATCHING

WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA

WHY IS LIFE SO BORING

WHY ARE CIGARETTES LEGAL

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WHY IS JESUS WHITE

WHY IS THERE LIQUID IN MY EAR

WHY DO Q TIPS FEEL GOOD

WHY DO GOOD PEOPLE DIE



WHY ARE ULTRASOUNDS IMPORTANT
WHY ARE ULTRASOUND MACHINES EXPENSIVE
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WHY ARE THERE SO MANY SPIDERS IN MY ROOM

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WHY IS DYING SO SCARY

WHY IS THERE NO GPS IN LAPTOPS

WHY DO KNEES CLICK

WHY AREN'T THERE E GRADES
WHY IS ISOLATION BAD
WHY DO BOYS LIKE ME
WHY DON'T BOYS LIKE ME
WHY IS THERE ALWAYS A JAVA UPDATE
WHY ARE THERE RED DOTS ON MY THIGHS

WHY IS SEX SO IMPORTANT



WHY DO IGUANAS DIE
WHY AREN'T THERE DINOSAUR GHOSTS

WHY ARE THERE FEMALE MR NIMES

WHY ARE THERE SWARMS OF GNATS

WHY IS THERE PHLEGM

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Methods of Point Estimation

- We will cover two popular methodologies to create point estimates of a population parameter.
 - Method of moments
 - Method of maximum likelihood
- Each approach can be used to create estimators with varying degrees of biasedness and relative MSE efficiencies.

Method of moments for point estimation

What are moments?

- The p-th **population moment** of a random variable is the expected value of X^p
 - First moment: $\mu = \int_{-\infty}^{+\infty} x f(x) dx$
 - Second moment: $\mu^2 + \sigma^2 = \int_{-\infty}^{+\infty} x^2 f(x) dx$
 - p-th moment: $\int_{-\infty}^{+\infty} x^p f(x) dx$
 - The **population moment** relates to the entire population
- A **sample moment** is calculated like its population moments but for a finite sample
 - Sample first moment = sample mean = $\frac{1}{n} \sum_{i=1}^n x_i$
 - Sample p-th moment $\frac{1}{n} \sum_{i=1}^n x_i^p$

Moment Estimators

Let X_1, X_2, \dots, X_n be a random sample from either a probability mass function or a probability density function with p unknown parameters $\theta_1, \theta_2, \dots, \theta_p$.

The **moment estimators** $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_p$ are found by equating the first p population moments to the first p sample moments and solving the resulting simultaneous equations for the unknown parameters.

Exponential Distribution: Moment Estimator-1st moment

- Suppose that x_1, x_2, \dots, x_n is a random sample from an exponential distribution $f(x) = \lambda \exp(-\lambda x)$ with parameter λ .
- There is only one parameter to estimate, so equating population and sample first moments, we have one equation: $E(X) = \bar{x}$.
- $E(X) = 1/\lambda$ thus $\lambda = 1/\bar{x}$ is the 1st moment estimator.

Matlab exercise

- Generate 100,000 exponentially distributed random numbers with $\lambda=3$: $f(x)=\lambda\exp(-\lambda x)$
 - Use `random('Exponential'...)` but read the manual to know how to introduce parameters.
- Get a moment estimate of lambda based on the 1st moment
- Get a moment estimate of lambda based on the 2nd moment
 - Second moment of the exponential distribution is $E(X^2) = E(X)^2 + \text{Var}(X) = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2$
- Get a moment estimate of lambda based on the 20th moment
 - Generally, p-th moment of the exponential distribution is $E(X^p) = p!/\lambda^p$

How I solved it

- Stats=100000;
- Y=random('Exponential', 1/3, Stats, 1);
%parametrization in MATLAB is 1/lambda
- 1/mean(Y) %matching the first moment
% ans = 3.0086
- sqrt(2/mean(Y.^2)) %matching the second moment
% ans = 3.0081
- (factorial(20)/mean(Y.^20))^(1./20) %matching the 20th moment

Credit: XKCD
comics

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Method of Maximum Likelihood for point estimation

Maximum Likelihood Estimators

- Suppose that X is a random variable with probability distribution $f(x, \theta)$, where θ is a single unknown parameter. Let x_1, x_2, \dots, x_n be the observed values in a random sample of size n . Then the **likelihood function** of the sample is the probability to get it in a random variable with PDF $f(x, \theta)$:

$$L(\theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdot \dots \cdot f(x_n, \theta) \quad (7-9)$$

- Note that the likelihood function is now a function of only the unknown parameter θ . The **maximum likelihood estimator** (MLE) of θ is the value of θ that maximizes the likelihood function $L(\theta)$.
- Usually, it is easier to work with **logarithms**: $l(\theta) = \ln L(\theta)$

Exponential MLF:

$$f(x_i) = \lambda e^{-\lambda x_i}$$

$$L(\lambda) = P(x_1, x_2, \dots, x_n | \lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i} =$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^n x_i}$$

$$\ln L(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^n x_i$$

$$\frac{d \ln L(\lambda)}{d \lambda} = \frac{n}{\lambda} - \sum x_i = 0$$

$$\hat{\lambda} = \frac{n}{\sum x_i} = \frac{1}{\bar{X}}$$

Same as
1st moment
estimator

Example 7-11: Exponential MLE

Let X be an exponential random variable with parameter λ . The likelihood function of a random sample of size n is:

$$L(\lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^n x_i}$$

$$\ln L(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^n x_i$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i = 0$$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{X}} \quad (\text{same as moment estimator})$$

Bernoulli: MLE

$$f(x, p) = p^x (1-p)^{1-x}$$

$$L(p) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} =$$

$$= p^{\sum x_i} (1-p)^{n - \sum x_i}$$

$$\ln L(p) = (\sum x_i) \ln p + (n - \sum x_i) \ln(1-p)$$

$$\frac{d \ln L(p)}{dp} = \frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p} = 0 \quad \text{at } \hat{p}$$

$$0 = \frac{(1 - \hat{p}) \sum x_i - \hat{p} (n - \sum x_i)}{\hat{p} (1 - \hat{p})} \quad \hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

Example 7-9: Bernoulli MLE

Let X be a Bernoulli random variable. The probability mass function is $f(x;p) = p^x(1-p)^{1-x}$, $x = 0, 1$ where P is the parameter to be estimated. The likelihood function of a random sample of size n is:

$$\begin{aligned} L(p) &= p^{x_1}(1-p)^{1-x_1} \cdot p^{x_2}(1-p)^{1-x_2} \cdot \dots \cdot p^{x_n}(1-p)^{1-x_n} \\ &= \prod_{i=1}^n p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i} \end{aligned}$$

$$\ln L(p) = \left(\sum_{i=1}^n x_i \right) \ln p + \left(n - \sum_{i=1}^n x_i \right) \ln(1-p)$$

$$\frac{d \ln L(p)}{dp} = \frac{\sum_{i=1}^n x_i}{p} - \frac{(n - \sum_{i=1}^n x_i)}{(1-p)} = 0$$

$$\hat{p} = \frac{\sum_{i=1}^n x_i}{n} \text{ (same as moment estimator)}$$

Normal MLE for μ

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$L(\mu, \sigma) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(-\frac{\sum (x_i - \mu)^2}{2\sigma^2}\right)$$

$$\ln L(\mu, \sigma) = -n \ln(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

$$\frac{d \ln L(\mu, \sigma)}{d\mu} = \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = 0 \text{ at } \hat{\mu}$$
$$\hat{\mu} = \frac{\sum_{i=1}^n x_i}{n}$$

Example 7-10: Normal MLE for μ

Let X be a normal random variable with unknown mean μ and variance σ^2 . The likelihood function of a random sample of size n is:

$$\begin{aligned}L(\mu) &= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\&= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i-\mu)^2} \\ \ln L(\mu) &= \frac{-n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \\ \frac{d \ln L(\mu)}{d\mu} &= \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = 0 \\ \hat{\mu} &= \frac{\sum_{i=1}^n x_i}{n} = \bar{X} \text{ (same as moment estimator)}\end{aligned}$$

Example 7-11: Normal MLE for σ^2

Let X be a normal random variable with the estimate of mean μ determined by MLE (see the previous slide) and an **unknown variance σ^2** . The likelihood function of a random sample of size n is:

$$\begin{aligned}L(\sigma) &= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{(2\sigma^2)}} \\&= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i-\mu)^2} \\ \ln L(\sigma) &= \frac{-n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \\ \frac{d \ln L(\sigma)}{d\sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - \mu)^2 = 0 \\ \widehat{\sigma^2} &= \frac{\sum_{i=1}^n (x_i - \mu)^2}{n} \quad (\text{biased estimator})\end{aligned}$$

MLE for Poisson distribution

$$\begin{aligned} f(x_1, \dots, x_n | \lambda) &= \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \dots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!} \\ &= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{x_1! \dots x_n!} \end{aligned}$$

$$\log f(x_1, \dots, x_n | \lambda) = -n\lambda + \sum_1^n x_i \log \lambda - \log c$$

where $c = \prod_{i=1}^n x_i!$ does not depend on λ , and

$$\frac{d}{d\lambda} \log f(x_1, \dots, x_n | \lambda) = -n + \frac{\sum_1^n x_i}{\lambda}$$

By equating to zero, we obtain that the maximum likelihood estimate $\hat{\lambda}$ equals

$$\hat{\lambda} = \frac{\sum_1^n x_i}{n}$$

Credit: XKCD
comics

WHY ARE THERE SLAVES IN THE BIBLE

WHY DO TWINS HAVE DIFFERENT FINGERPRINTS
WHY ARE AMERICANS AFRAID OF DRAGONS

WHY IS HTTPS CROSSED OUT IN RED
WHY IS THERE A LINE THROUGH HTTPS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT

QUESTIONS

FOUND IN GOOGLE AUTOCOMPLETE



WHY ARE THERE WEEKS
WHY DO I FEEL DIZZY

WHY AREN'T ECONOMISTS RICH

WHY DO AMERICANS CALL IT SOCCER

WHY ARE MY EARS RINGING

WHY ARE THERE SO MANY AVENGERS

WHY ARE THE AVENGERS FIGHTING THE X MEN
WHY IS WOLVERINE NOT IN THE AVENGERS

WHY ARE THERE ANTS IN MY LAPTOP

WHY IS EARTH TILTED

WHY IS SPACE BLACK
WHY IS OUTER SPACE SO COLD
WHY ARE THERE PYRAMIDS ON THE MOON
WHY IS NASA SHUTTING DOWN



WHY IS THERE AN OWL IN MY BACKYARD

WHY IS THERE AN OWL OUTSIDE MY WINDOW

WHY IS THERE AN OWL ON THE DOLLAR BILL

WHY DO OWLS ATTACK PEOPLE

WHY ARE AK 47s SO EXPENSIVE

WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE

WHY ARE THERE GODS

WHY ARE THERE TWO SPOCKS

WHY IS MT VESUVIUS THERE

WHY DO THEY SAY T MINUS

WHY ARE THERE OBELISKS

WHY ARE WRESTLERS ALWAYS WET

WHY ARE OCEANS BECOMING MORE ACIDIC

WHY IS ARWEN DYING

WHY AREN'T MY QUAIL LAYING EGGS
WHY AREN'T MY QUAIL EGGS HATCHING

WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA

WHY ARE CIGARETTES LEGAL

WHY ARE THERE DUCKS IN MY POOL

WHY IS JESUS WHITE

WHY IS THERE LIQUID IN MY EAR

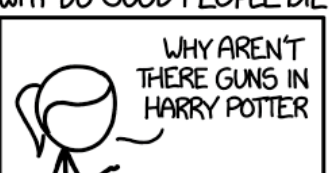
WHY DO Q TIPS FEEL GOOD

WHY DO GOOD PEOPLE DIE

WHY ARE ULTRASOUNDS IMPORTANT

WHY ARE ULTRASOUND MACHINES EXPENSIVE

WHY IS STEALING WRONG



WHY ARE DOGS AFRAID OF FIREWORKS
WHY IS THERE NO KING IN ENGLAND

WHY DO WHALES JUMP
WHY ARE WITCHES GREEN
WHY ARE THERE MIRRORS ABOVE BEDS

WHY DO I SAY UH

WHY IS SEA SALT BETTER

WHY ARE THERE TREES IN THE MIDDLE OF FIELDS

WHY IS THERE NOT A POKEMON MMO

WHY IS THERE LAUGHING IN TV SHOWS

WHY ARE THERE DOORS ON THE FREEWAY

WHY ARE THERE SO MANY SVCHOST.EXE RUNNING

WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA

WHY ARE THERE SCARY SOUNDS IN MINECRAFT

WHY IS THERE KICKING IN MY STOMACH

WHY ARE THERE TWO SLASHES AFTER HTTP

WHY ARE THERE CELEBRITIES

WHY DO SNAKES EXIST

WHY DO OYSTERS HAVE PEARLS

WHY ARE DUCKS CALLED DUCKS

WHY DO THEY CALL IT THE CLAP

WHY ARE KYLE AND CARTMAN FRIENDS

WHY IS THERE AN ARROW ON AANG'S HEAD

WHY ARE TEXT MESSAGES BLUE

WHY ARE THERE MUSTACHES ON CLOTHES

WHY ARE THERE MUSTACHES ON CARS

WHY ARE THERE MUSTACHES EVERYWHERE

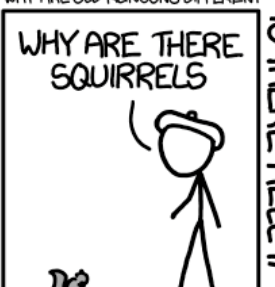
WHY ARE THERE SO MANY BIRDS IN OHIO

WHY IS THERE SO MUCH RAIN IN OHIO

WHY IS OHIO WEATHER SO WEIRD

WHY ARE THERE MALE AND FEMALE BIKES

WHY ARE THERE BRIDESMAIDS
WHY DO DYING PEOPLE REACH UP
WHY AREN'T THERE VARICOSE ARTERIES
WHY ARE OLD KUNGONS DIFFERENT



WHY IS PROGRAMMING SO HARD

WHY IS THERE A 0 OHM RESISTOR

WHY DO AMERICANS HATE SOCCER

WHY DO RHYMES SOUND GOOD

WHY DO TREES DIE

WHY ARE THERE TINY SPIDERS IN MY HOUSE

WHY DO SPIDERS COME INSIDE

WHY ARE THERE HUGE SPIDERS IN MY HOUSE

WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE

WHY ARE THERE SPIDERS IN MY ROOM

WHY ARE THERE SO MANY SPIDERS IN MY ROOM

WHY DO SPIDER BITES ITCH

WHY IS DYING SO SCARY

WHY IS THERE NO GPS IN LAPTOPS

WHY DO KNEES CLICK

WHY AREN'T THERE E GRADES

WHY IS SEX SO IMPORTANT

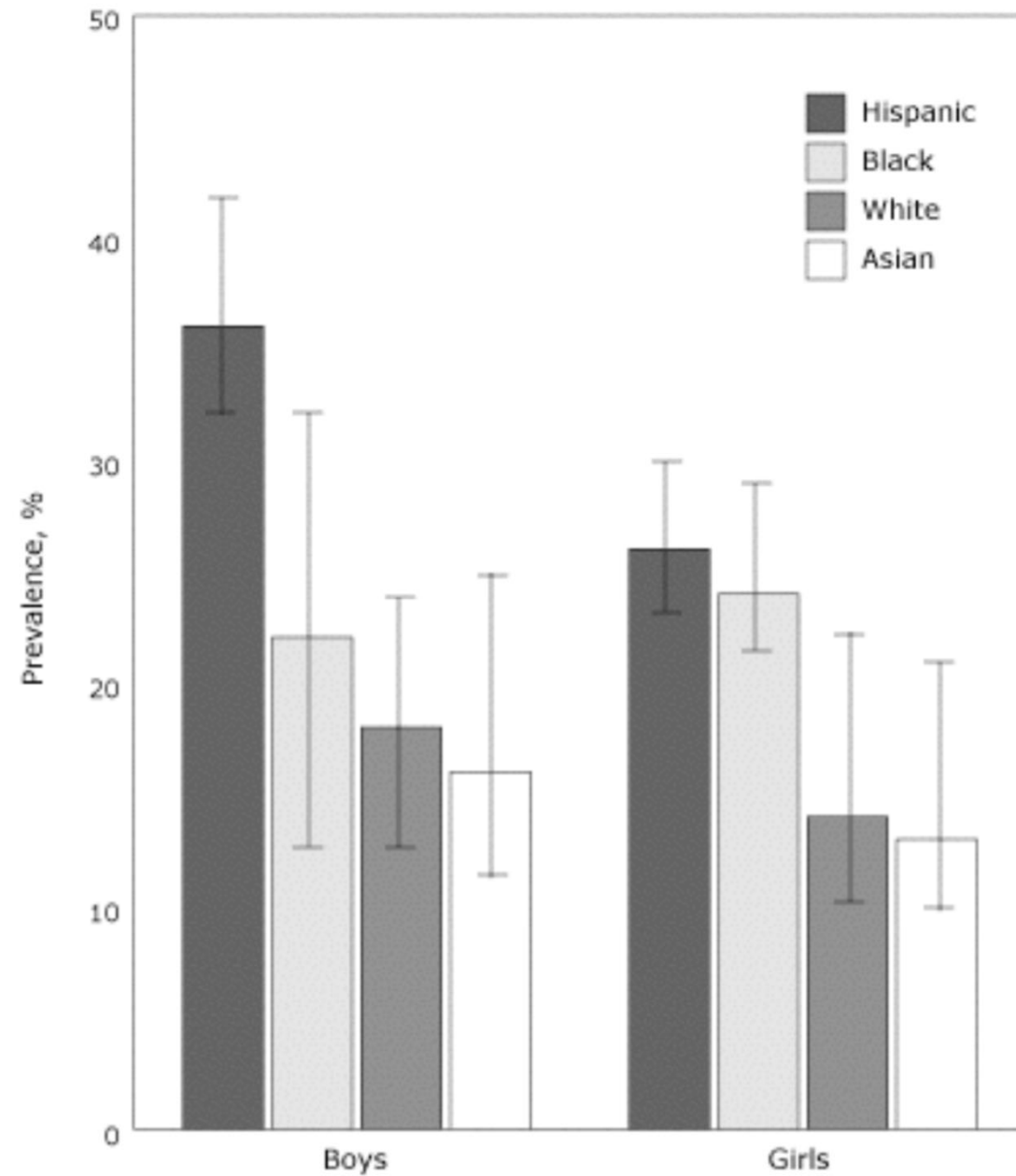


WHY AREN'T THERE DINOSAUR GHOSTS

WHY ARE THERE FEMALE MR NIMES

WHY IS GPS FREE

Confidence Intervals



Prevalence (with 95% CI bars) of obesity among New York City public elementary schoolchildren, by sex and race/ethnicity, 2003.

(source: CDC.GOV)

What do those bars actually mean?

ARTICLES

Patterns of somatic mutation in human cancer genomes

What does confidence interval mean?

The numbers of passenger and driver mutations present can be estimated from these results (see Supplementary Methods). Of the 921 base substitutions in the primary screen, 763 (95% confidence interval, 675–858) are estimated to be passenger mutations. Therefore, the large majority of mutations found through sequencing cancer genomes are not implicated in cancer development, even when the search has been targeted to the coding regions of a gene family of high candidature. However, there are an estimated 158 driver mutations (95% confidence interval, 63–246), accounting for the observed positive selection pressure. These are estimated to be distributed in 119 genes (95% confidence interval, 52–149). The number of samples containing a driver mutation is estimated to be 66 (95% confidence interval, 36–77). The results, therefore, provide statistical evidence for a large set of mutated protein kinase genes implicated in the development of about one-third of the cancers studied.

- We have talked about how a parameter can be estimated from sample data. However, it is important to understand how good is the estimate obtained.
- Bounds that represent an interval of plausible values for a parameter are an example of an **interval estimate**.

Two-sided confidence intervals

- Calculated based on the sample X_1, X_2, \dots, X_n
- Characterized by:
 - lower- and upper- confidence limits L and R
 - the confidence coefficient $1-\alpha$
- Objective: for two-sided confidence interval, find L and R such that
 - $\text{Prob}(\mu > R) = \alpha/2$
 - $\text{Prob}(\mu < L) = \alpha/2$
 - Therefore, $\text{Prob}(L < \mu < R) = 1-\alpha$
- For one-sided confidence interval, say, upper bound of μ , find R that
 - $\text{Prob}(\mu > R) = \alpha$
- **Assume standard deviation sigma is known**

Consider $1 - \alpha = 95\% = 0.95$

$$\alpha = 0.05; \quad \frac{\alpha}{2} = 0.025$$

$$z_{\alpha/2} = 1.96 \rightarrow \text{Prob}(Z > z_{\alpha/2}) = \frac{\alpha}{2}$$



$$\text{Prob}\left(-z_{\frac{\alpha}{2}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\text{Prob}\left(\bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

For one sided lower bound on μ

$$\text{Prob}\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \underline{z_{\alpha}}\right) \rightarrow$$

$$\mu > \bar{X} - z_{\alpha} \frac{\sigma}{\sqrt{n}}$$

$$z_{\alpha} = 1.65 <$$

$$z_{\alpha/2} = 1.96$$

Exercise

Ishikawa et al. (Journal of Bioscience and Bioengineering 2012) studied the force with which bacterial biofilms adhere to a solid surface.

Five measurements for a bacterial strain of *Acinetobacter* gave readings 2.69, 5.76, 2.67, 1.62, and 4.12 dyne-cm².

Assume that the standard deviation is known to be 0.66 dyne-cm²

- (a) Find 95% confidence interval for the mean adhesion force
- (b) If scientists want the width of the confidence interval to be below 0.55 dyne-cm² what number of samples should be?

Ishikawa et al. (Journal of Bioscience and Bioengineering 2012) studied the force with which bacterial biofilms adhere to a solid surface. Five measurements for a bacterial strain of Acinetobacter gave readings 2.69, 5.76, 2.67, 1.62, and 4.12 dyne-cm². Assume that the **standard deviation is known to be 0.66 dyne-cm²**

- (a) Find 95% confidence interval for the mean adhesion force
- (b) If scientists want the width of the confidence interval to be below 0.55 dyne-cm² what number of samples should be?

a) 95% CI for μ , $n = 5$ $\sigma = 0.66$ $\bar{x} = 3.372$, $z = 1.96$

$$\bar{x} - z\sigma / \sqrt{n} \leq \mu \leq \bar{x} + z\sigma / \sqrt{n}$$

$$3.372 - 1.96(0.66 / \sqrt{5}) \leq \mu \leq 3.372 + 1.96(0.66 / \sqrt{5})$$

$$2.79 \leq \mu \leq 3.95$$

b) Width is $2z\sigma / \sqrt{n} = 0.55$, therefore $n = [2z\sigma / 0.55]^2 = [2(1.96)(0.66) / 0.55]^2 = 22.13$
Round up to $n = 23$.