Homework #3

Please present 4 significant figures in your final answers for probabilities. Also, make sure to explain your thought process as if the reader is one of your classmates.

1. **(20 points)** The joint probability mass function of discrete random variables X and Y taking values x = 1, 2, 3 and y = 1, 2, 3, respectively, is given by a formula $f_{XY}(x, y) = c*(x + 2*y)$. Determine the following:

a) **(2 points)** Find c

Answer: $\sum_R f(x,y) = c*(3+5+7+4+6+8+5+7+9) = 1$, c*54 = 1. Thus, c = 1/54

b) **(2 points)** Find probability of the event where X = 1 and Y < 3

Answer: $P(X = 1, Y < 3) = f_{XY}(1,1) + f_{XY}(1,2) = \frac{1}{36}(3+5) = 8/54$

c) **(2 points)** Find marginal probability $P_Y(Y = 2)$

Answers: $P(Y = 2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \frac{5+6+7}{54} = 18/54$

d) **(2 points)** Marginal probability distribution of the random variable X

Answers: marginal distribution of *X*

x	$f_X(x) = f_{XY}(x,1) + f_{XY}(x,2) + f_{XY}(x,3)$
1	1/54 (3 + 5 + 7) = 0.2778
2	1/54 (4 + 6 + 8) = 0.3333
3	1/54 (5 + 7 + 9) = 0.3889

(e) **(2 points)** Marginal probability distribution of the random variable Y

Answers: marginal distribution of *Y*

y	$f_Y(y) = f_{XY}(1,y) + f_{XY}(2,y) + f_{XY}(3,y)$
1	1/54 (3 + 4 + 5) = 0.2222
2	1/54 (5 + 6 + 7) = 0.3333
3	1/54 (7 + 8 + 9) = 0.4444

(f) **(4 points)** E(X), E(Y), V(X), and V(Y)

Answers:

$E(X) = (1 \times 0.2778) + (2 \times 0.3333) + (3 \times 0.3889) = 2.111$

$V(X) = 0.2778 * (1 - 2.167)^2 + 0.3333 * (2 - 2.167)^2 + 0.3889 * (3 - 2.167)^2 = 0.6575$

$E(Y) = (1 \times 0.2222) + (2 \times 0.3333) + (3 \times 0.4444) = 2.222$

$V(X) = 0.2222 * (1 - 2.222)^2 + 0.3333 * (2 - 2.222)^2 + 0.4444 * (3 - 2.222)^2 = 0.6172$

(g) **(2 points)** Find conditional probability distribution of Y given that X = 1

Answers: $f_{Y|X}(y) = \dfrac{f_{XY}(1,y)}{f_X(1)}$

y	$f_{Y\|X}(y)$
1	(3/54)/(0.2778)=0.20
2	(5/54)/(0.2778)=0.3333
3	(7/54)/(0.2778)=0.4667

(h) **(2 points)** Conditional probability distribution of X given that Y = 2

Answers: $f_{X|Y}(x) = \dfrac{f_{XY}(x,2)}{f_Y(2)}$ and $f_Y(2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = (5 + 6 + 7)/54 =$ 0.3333

x	$f_{X\|Y}(x)$
1	(5/54)/(0.3333)=0.2778
2	(6/54)/(0.3333)=0.3334
3	(7/54)/(0.3333)=0.3889

(i) **(2 points)** Are X and Y independent?

Answers: No. $f_{XY}(x,y) \mathrel{!}= f_X(x) * f_Y(y)$

2. **(6 points)** A random variable X has density function $f(X = x) = c(x + x^{0.5})$ for $x \in [0,1]$ and $f(X = x) = 0$ otherwise.

a) **(3 points)** Determine c.

Answer: $c \int_0^1 x + x^{0.5} dx = 1$

$c\left(\frac{7}{6}\right) = 1$

c = 6/7.

b) **(3 points)** Compute E(1/X)

Answer: E(1/X) = $\int_0^1 \left(\frac{1}{x}\right) f(x)\, dx = \int_0^1 \left(\frac{1}{x}\right) (\frac{6}{7}(x + x^{0.5}))\, dx = 2.571$

3. **(10 points)** Toss an unfair coin 4 times. The probability of head is 0.6 and that of tail is 0.4. Let X be the total number of heads among the first two tosses and Y the total number of heads among the last two tosses.

a) **(4 points)** Write down the joint probability mass fraction of X and Y.

Answers:

x/y	0	1	2	Margin
0	$0.4^4 = 0.0256$	$0.4^2 \times 2 \times 0.6 \times 0.4 = 0.0768$	$0.4^2 \times 0.6^2 = 0.0576$	0.16
1	$2 \times 0.6 \times 0.4 \times 0.4^2 = 0.0768$	$2 * 0.4 * 0.6 \times 2 \times 0.6 \times 0.4 = 0.2304$	$2 \times 0.6 \times 0.4 \times 0.6^2 = 0.1728$	0.48
2	$0.6^2 \times 0.4^2 = 0.0576$	$0.6^2 \times 2 \times 0.6 \times 0.4 = 0.1728$	$0.6^4 = 0.1296$	0.36
Margin	0.16	0.48	0.36	1.00

b) **(2 points)** Are X and Y independent? Please explain.
Answers: Independent.

c) **(4 points)** Compute the conditional probability $P(X > Y|Y = 0)$
Answers: $P(X > Y|Y = 0) = \frac{P(X \geq Y, Y=0)}{P(Y=0)} = \frac{P(X=1,Y=0)+P(X=2,Y=0)}{0.16} = \frac{0.0768+0.0576}{0.16} = 0.84$

4. (7 points) A train has 200 seats. Suppose that the mean weight of men is 180 pounds with a standard deviation of 20 and the mean weight of women 170 is with a standard deviation of 15. Assume that all the seats in trains are always occupied, there are always 100 men and 100 women on the train and that the weight of different passengers are independent random variables.

a) **(4 points)** Find mean and standard deviation of the mean weight of people on the train.

Answers: $A = \frac{1}{200}(\sum_i^{100} X_m + \sum_j^{100} X_w)$

$$E(A) = \frac{1}{200} E(\sum_i^{100} X_m + \sum_j^{100} X_w) = \frac{1}{200}\left(100 * E(X_m) + 100 * E(X_w)\right)$$
$$= \frac{1}{200}(100 * 180 + 100 * 170) = 175,$$

$$V(A) = V(\frac{1}{200}\left(\sum_i^{100} X_m + \sum_j^{100} X_w)\right) = \frac{1}{200^2}\left(\sum_i^{100} V(X_m) + \sum_i^{100} V(X_w)\right)$$
$$= \frac{1}{200^2}\left(100 * V(X_m) + 100 * V(X_w)\right) = \frac{1}{200^2}(100 * 20^2 + 100 * 15^2)$$
$$= 1.5625$$

So, $\sigma_a = \sqrt{1.5625} = 1.25$

b) **(3 points)** Find the probability that when you randomly pick a train, the mean weight of people on it is greater than 176 pounds.

Answers: Using the Central Limit Theorem, we can claim that

$$W \sim Normal(\mu_a = 175, \sigma_a = 1.25)$$

P(W>174) = P(Z>(176-175)/1.25) = P(Z>0.8) = 0.21186

5. **(4 points)** Suppose random variables X, Y have standard derivations, $\sigma_X = 3$ and $\sigma_Y = 4$, respectively, and correlation coefficient corr(X, Y) = −0.5.

(a) **(2 points)** Find cov(X, Y).

Answer: $cov(X, Y) = Corr(X, Y) * \sigma_X * \sigma_Y = -6$

(b) **(2 points)** Find Var(5X − 3Y).

Answers:

$$Var(5X - 3Y) = 25 * Var(X) + 9 * Var(Y) - 2 * 5 * 3 * Cov(X, Y)$$
$$= 25 * 9 + 9 * 16 - 30 * (-6) = 549$$

6. **(12 points)** Time interval separating subsequent bus arrivals at a stop is an exponential random variable with mean 20 minutes. Steve and Andrew work at the same place and each will be late to work unless they board a bus on or before 8:50am. Steve comes to the bus stop exactly at 8am. Andrew also comes to the same bus stop but at a random time, uniformly distributed between 8am and 8:30am. Both of them take the first bus that arrives.

(a) **(4 points)** What is the probability that Steve will be late for work tomorrow?

Answers: $P(Steve \text{ late}) = 1 - P(T < 50) = 1 - \frac{1}{20}\int_0^{50} e^{-\frac{t}{20}} dt = 1 - 0.91791 = 0.08208$

(b) **(4 points)** What is the probability that Andrew will be late for work tomorrow?

Answers:

$$P(Andrew\ \text{ late}) = \int_0^{30} \frac{dx}{30} P(T >= 50 | T > x) = \int_0^{30} \frac{dx}{30} e^{-(50-x)/20} = \frac{e^{-2.5}}{30} \int_0^{30} e^{x/20}\, dx$$
$$= \frac{20e^{-2.5}}{30}\left(e^{3/2} - 1\right) = 0.1905$$

(c) **(4 points)** What is the probability that Steve and Andrew will different busses?

Probability that Steve will not leave by the time x when Andrew comes is exp(-x/20).

It needs to be integrated over Int_0^30 dx/30 exp(-x/20)=

Answers: 1-$P(\text{Steve}\ \text{ and Andrew meet}) = 1 - \int_0^{30} \frac{dx}{30} e^{-\frac{x}{20}} = 1 - \frac{20}{30}\left(1 - e^{-\frac{30}{20}}\right) = 0.4821$