Inductive probability
relies on combinatorics
or the art of counting
combinations

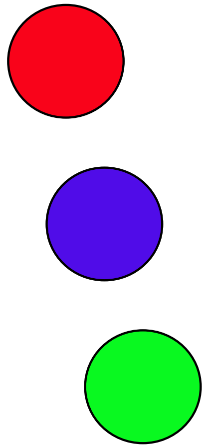
Multiplication and permutation rules are two examples of a general problem, where a sample of size k is drawn from a population of n distinct objects

Balls drawn from an urn (or a bag)

1 ball is red

1 ball is blue

1 ball is green

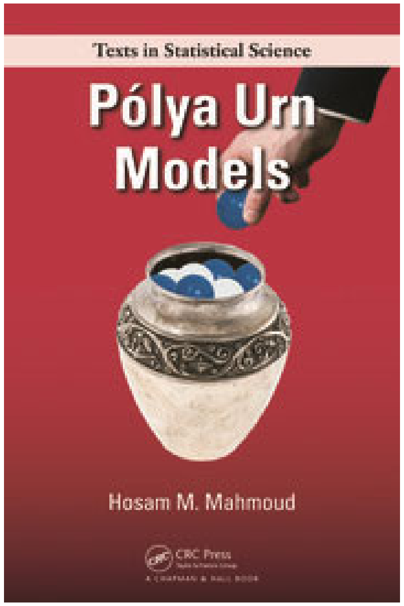


n=3 balls of different colors in a bag from which I draw k=2 balls one at a time

- Do I put each ball back to the bag after drawing it?
  - Yes: problem with replacement
  - No: problem without replacement
- Do I keep track of the order in which the balls were drawn?
  - Yes: the order matters
  - No: the order does not matter

George Pólya

- George Pólya (December 13, 1887 – September 7, 1985) was a Hungarian mathematician.
- He was a professor of mathematics at ETH Zürich from 1914 to 1940 and at Stanford University from 1940 to 1953. He made fundamental contributions to combinatorics, number theory, numerical analysis and probability theory.

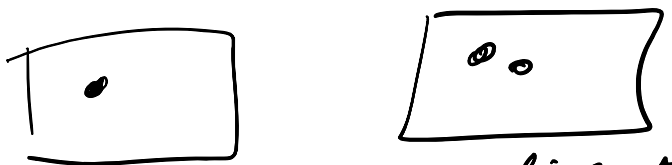


How many ways to choose a sample of K objects out of a population of n objects

	Order matters	Order does not matter
Replace	$n \times n \times n \times \ldots n = n^K$	~~$\frac{n^K}{K!}$~~ not all objects are different
Do not replace	$n \times (n-1) \times (n-2) \times \ldots \times (n-K+1) = \frac{n!}{(n-K)!}$	All objects are different → $\frac{n!}{(n-K)!} \times \frac{1}{K!} = \binom{n}{K}$

How to solve the problem of K out of n with replacement but where order does not matter?

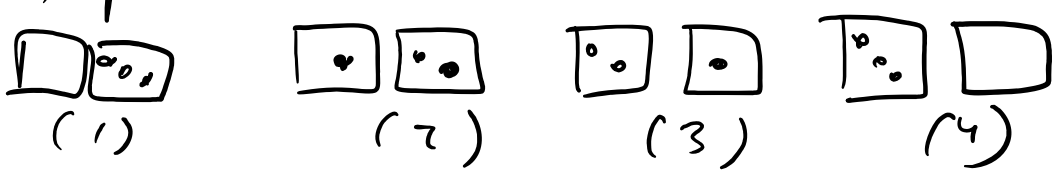
Let's solve n=2 problem first:

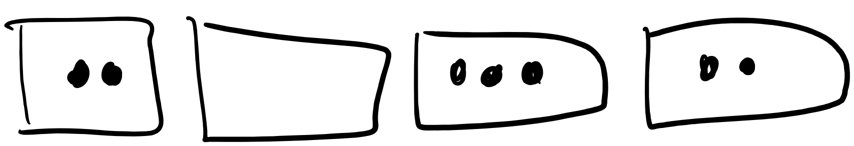


object 1 object 2

$K=3$

4 possibilities





$n=4,\ k=7$

••||•••|•• K dots, n−1 box boundaries

$$\binom{k+n-1}{k} = \frac{(k+n-1)!}{k!\,(n-1)!}$$ ways to distribute

Sampling table

How many ways to choose a sample of k objects out of population of n objects?

	Order matters	Order does not matter
Replacement	$(n)^k$	Difficult: $\binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!k!}$
No replacement	$n(n-1)(n-2)..(n-k+1) = \frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Example

- A DNA of 100 bases is characterized by its numbers of 4 nucleotides: d_A, d_C, d_G, and d_T ($d_A+d_C+d_G+d_T=100$)

- **I don't care about the sequence** (only about the total numbers of A,C,G, and T

- How many distinct combinations of d_A, d_C, d_G, and d_T are out there?

What is n and k in this problem?

A. k=100, n=4
B. k=4, n=100
C. k=100, $n=4^{100}$
D. $k=4^{100}$, n=4
E. I don't know

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What is n and k in this problem?

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D. $k=4^{100}$, n=4
E. I don't know

Is it sampling with or without replacement & does order matter?

A. with replacement, order matters
B. without replacement, any order
C. with replacement, any order
D. without replacement, order matters
E. I don't know

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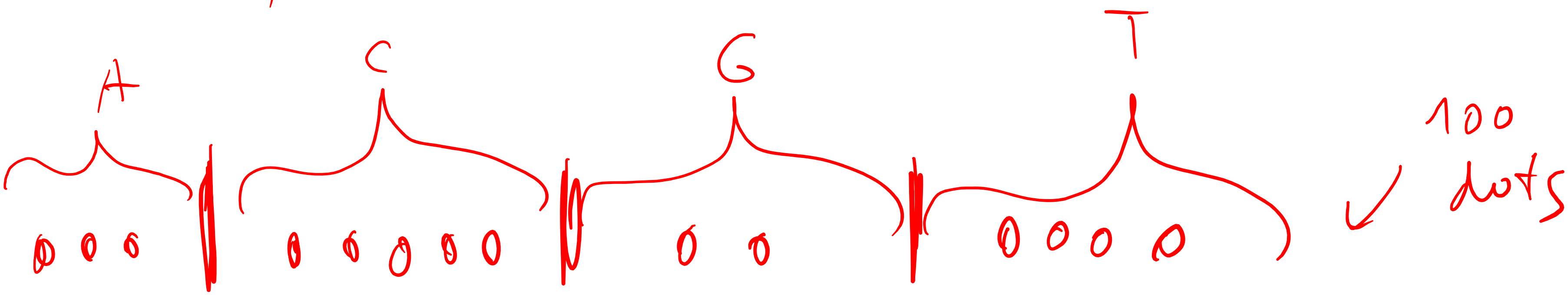
Is it sampling with or without replacement & does order matter?

A. with replacement, order matters
B. without replacement, any order
C. with replacement, any order
D. without replacement, order matters
E. I don't know

$n = 4, \ K = 100$



$$\# \text{ of possibilities} = \frac{(100+4-1)!}{(4-1)!\ 100!} = \frac{103 \cdot 102 \cdot 101}{3 \cdot 2 \cdot 1} = 176{,}851$$

If order did matter

$$4^{100} = 2^{200} = 10^{60} \approx \# \text{ atoms in a galaxy}$$

Probability Axioms, Conditional Probability, Statistical (In)dependence, Circuit Problems

Axioms of probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

(3) For two events E_1 and E_2 with $E_1 \cap E_2 = \varnothing$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

These axioms imply that:

$$P(\varnothing) = 0$$

$$P(E') = 1 - P(E)$$

if the event E_1 is contained in the event E_2

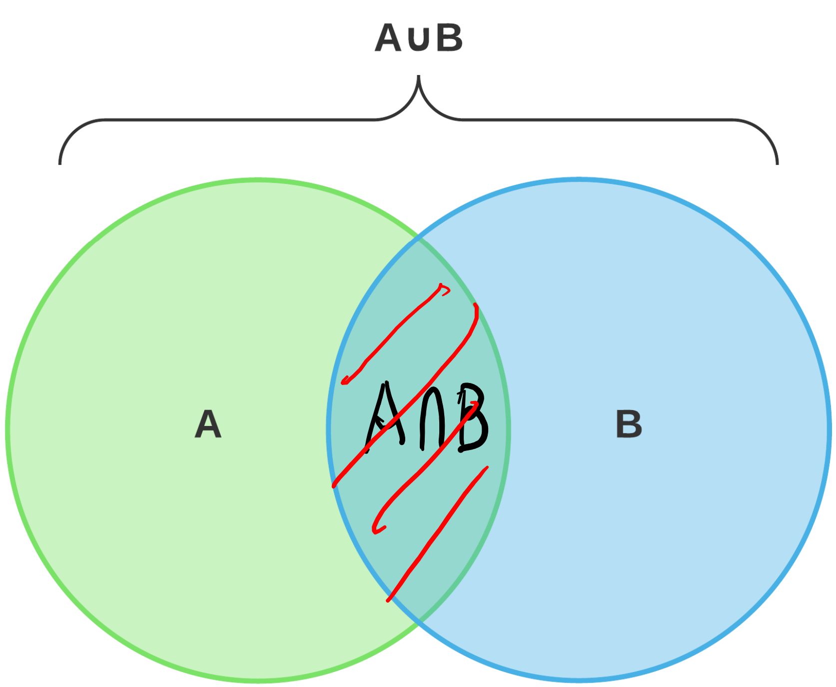
$$P(E_1) \leq P(E_2)$$

Addition rules following from the Axiom (3)

If A and B are mutually exclusive events, i.e. $A \cap B = \emptyset$

$$P(A \cup B) = P(A) + P(B) \qquad (2\text{-}2)$$

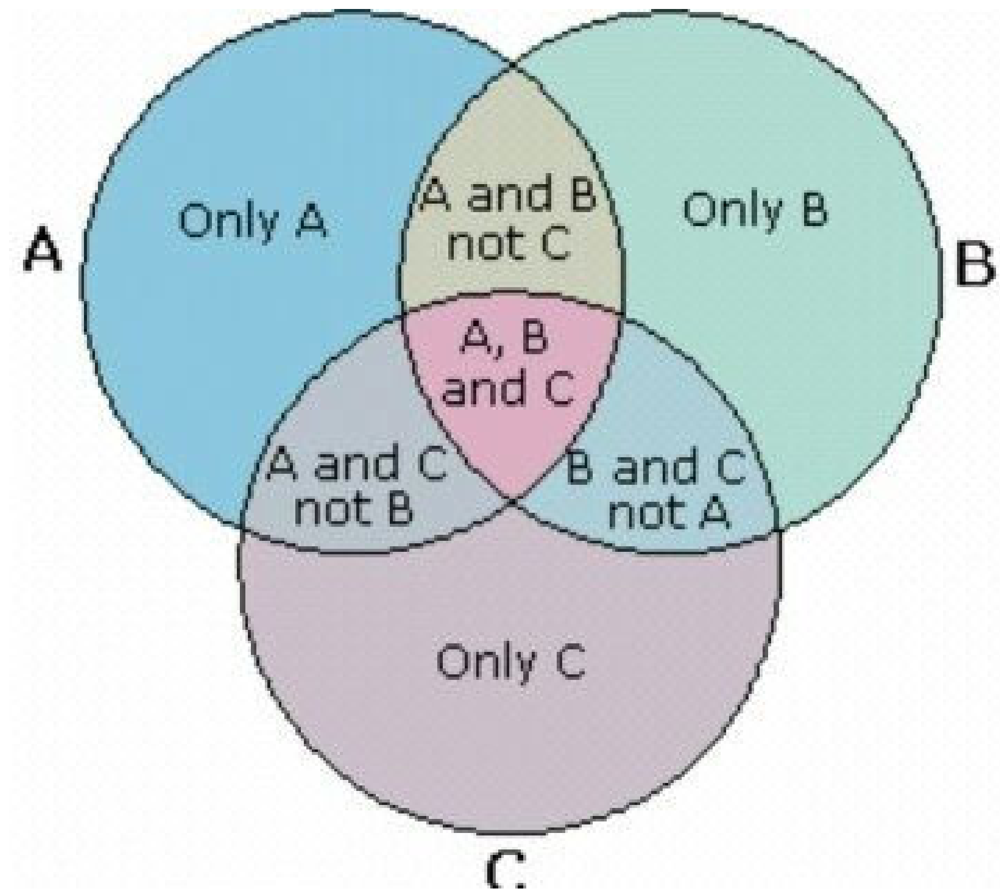
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad (2\text{-}1)$$



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) -$$

$$- P(A \cap B) - P(A \cap C) - P(B \cap C) +$$

$$+ P(A \cap B \cap C).$$



Conditional probability

The **conditional probability** of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = P(A \cap B)/P(A)$$

for $P(A) > 0$.

This definition can be understood in a special case in which all outcomes of a random experiment are equally likely. If there are n total outcomes,

$$P(A) = (\text{number of outcomes in } A)/n$$

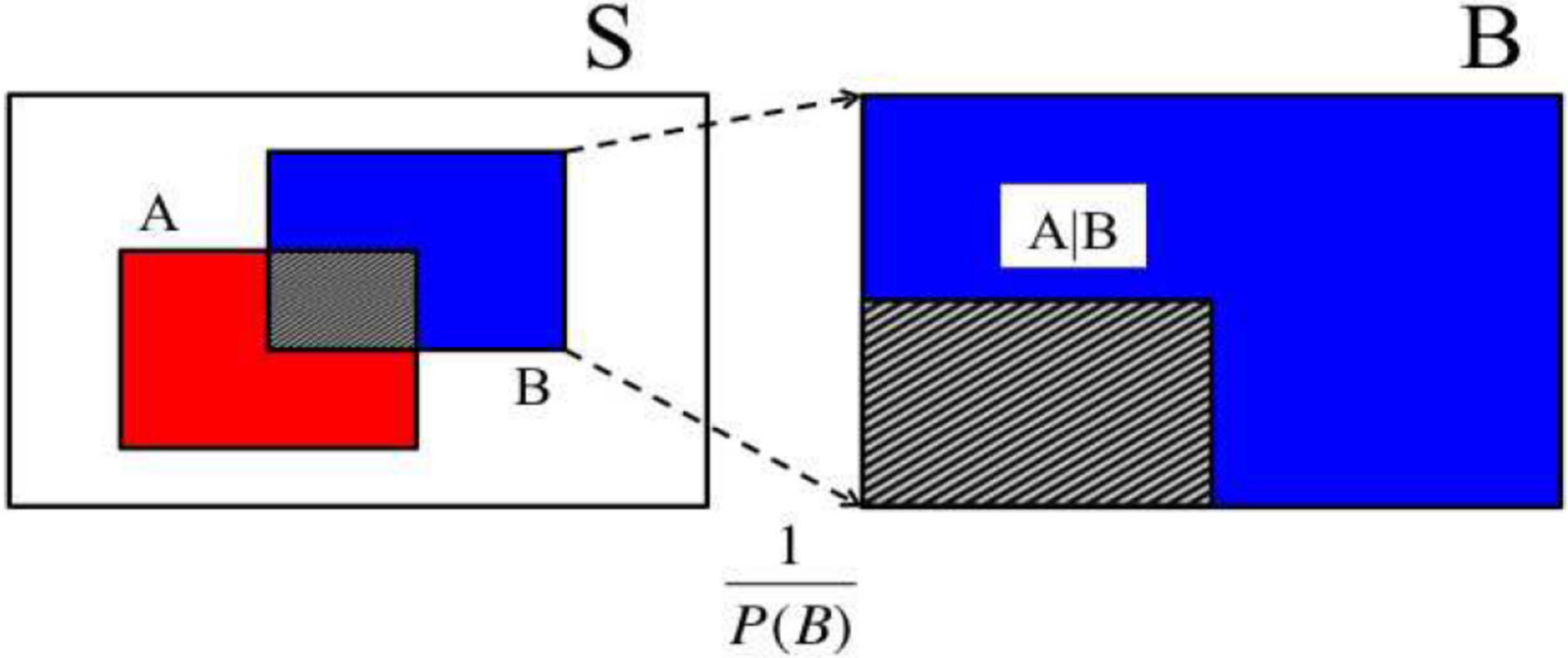
Also,

$$P(A \cap B) = (\text{number of outcomes in } A \cap B)/n$$

Consequently,

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

Therefore, $P(B|A)$ can be interpreted as the relative frequency of event B among the trials that produce an outcome in event A.



S
A
B
B
A|B
$\frac{1}{P(B)}$

Multiplication rule

is just definition of conditional probability

$P(B|A) = P(B \cap A)/P(A) \rightarrow$

$$P(B \cap A) = P(B|A) \cdot P(A)$$

Drake equation

$$N = R^{*} \cdot f_p \cdot n_e \cdot f_l \cdot f_i \cdot f_c \cdot L$$

- N = The number of civilizations in The Milky Way Galaxy whose electromagnetic emissions are detectable.
- R* = The rate of formation of stars suitable for the development of intelligent life.
- f_p = The fraction of those stars with planetary systems.
- n_e = The number of planets, per solar system, with an environment suitable for life.
- f_l = The fraction of suitable planets on which life actually appears.
- f_i = The fraction of life bearing planets on which intelligent life emerges.
- f_c = The fraction of civilizations that develop a technology that releases detectable signs of their existence into space.
- L = The length of time such civilizations release them

Ms. Perez figures that there is a 5% chance that her company will set up a branch in Phoenix. If it does, she is 10% certain that she will be made its manager. What is the probability that Perez will be a Phoenix branch office manager?

A. 15%
B. 0.5%
C. 50%
D. 5%
E. 10%

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Statistically independent events

Always true: $P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$

- **Two events**

Two events are **independent** if any one of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

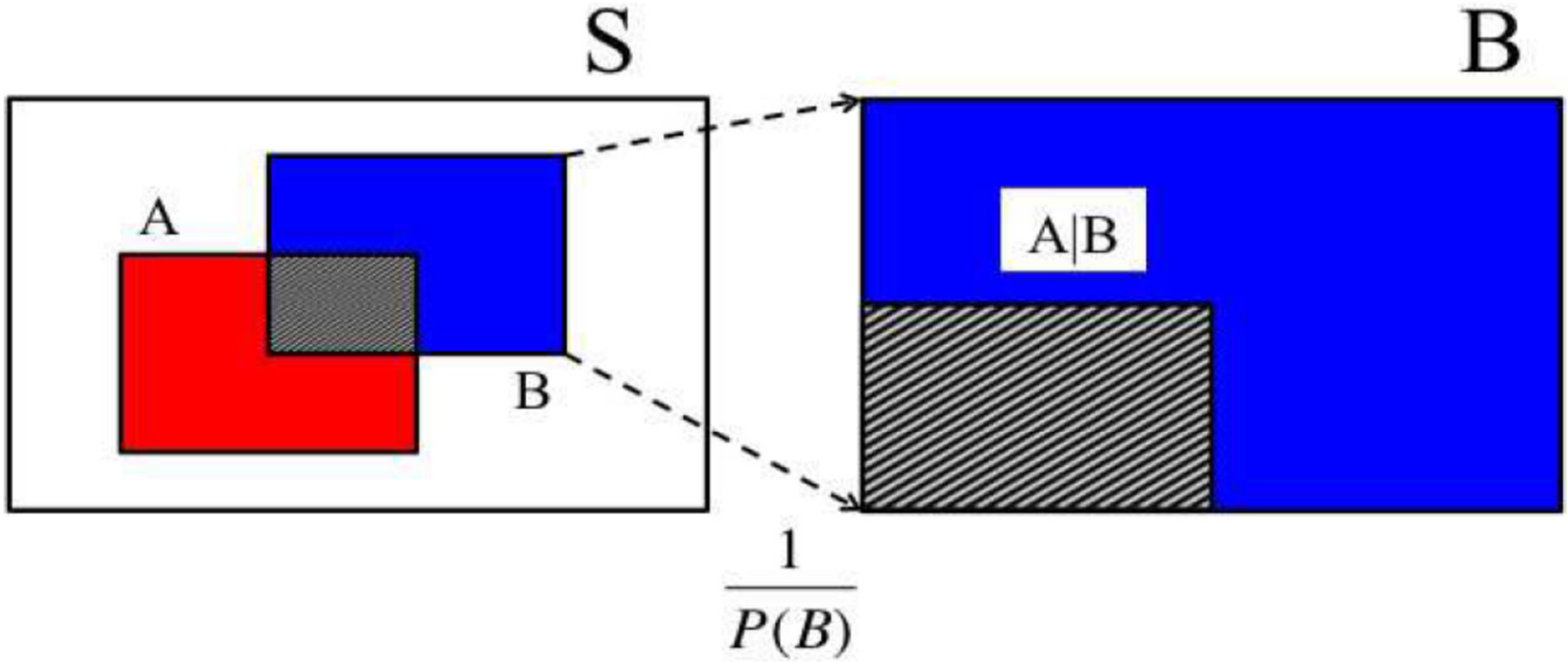
(2) $P(B|A) = P(B)$

(3) $P(A \cap B) = P(A)P(B)$

- **Multiple events**

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots \times P(E_{i_k})$$


S
B
A
B
A|B
$\frac{1}{P(B)}$

Example 3.10. Let an experiment consist of drawing a card at random from a standard deck of 52 playing cards. Define events A and B as "the card is a ♠" and "the card is a queen." Are the events A and B independent? By definition, $P(A \cdot B) = P(Q♠) = \frac{1}{52}$. This is the product of $P(♠) = \frac{13}{52}$ and $P(Q) = \frac{4}{52}$, and events A and B in question are independent. In this situation, intuition provides no help. Now, pretend that the 2♡ is drawn and excluded from the deck prior to the experiment. Events A and B become dependent since

$$\mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{13}{51} \cdot \frac{4}{51} \neq \frac{1}{51} = \mathbb{P}(A \cdot B).$$

Credit: XKCD comics