

Central Limit Theorem

If X_1, X_2, \dots, X_n is a random sample of size n is taken from a population with mean μ and **finite variance σ^2** , and **any distribution**. If \bar{X} is the sample mean, then the **limiting form of the distribution** of

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \quad (7-1)$$

for **large n** , is the **standard normal distribution**.

If X_1, X_2, \dots, X_n are themselves normally distributed – for any n

Test CLT for your own random variable

- Go to:
https://onlinestatbook.com/stat_sim/sampling_dist/
- Select “Custom” at the top and use mouse to sketch the PMF of your own random variable
- Select “mean” and $n=5$ in the third panel
- Choose “Animated” in the second panel and use `number_of_experiments=5` to see one sample being generated
- Repeat with `number_of_experiments =10,000`
- Now select “mean” and $n=25$ in the fourth panel
- Skewness and Kurtosis are measures of how good is the normal (Gaussian) fit (choose “fit normal”)

Sampling Distributions of Sample Means

Figure 7-1 Distributions of average scores from throwing dice.

$$\text{Mean} = (6+1)/2=3.5$$

$$\text{Sigma}^2 = [(6-1+1)^2-1]/12=2.92$$

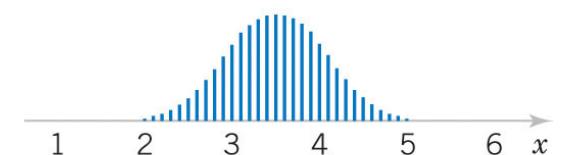
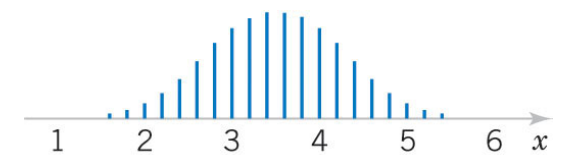
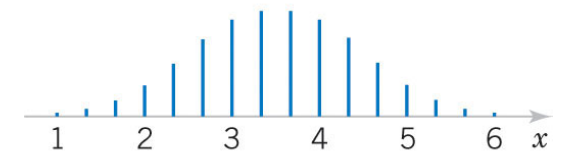
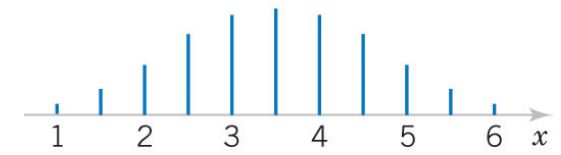
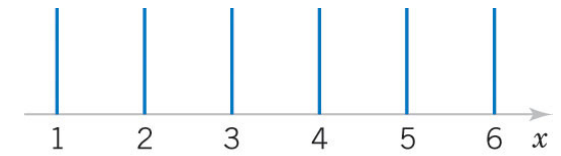
$$\text{Sigma}=1.71$$

Formulas

$$\mu = \frac{b+a}{2} = 3.5$$

$$\sigma_X^2 = \frac{(b-a+1)^2-1}{12} = 35/12$$

$$\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n}$$



show
Matlab

Matlab exercise

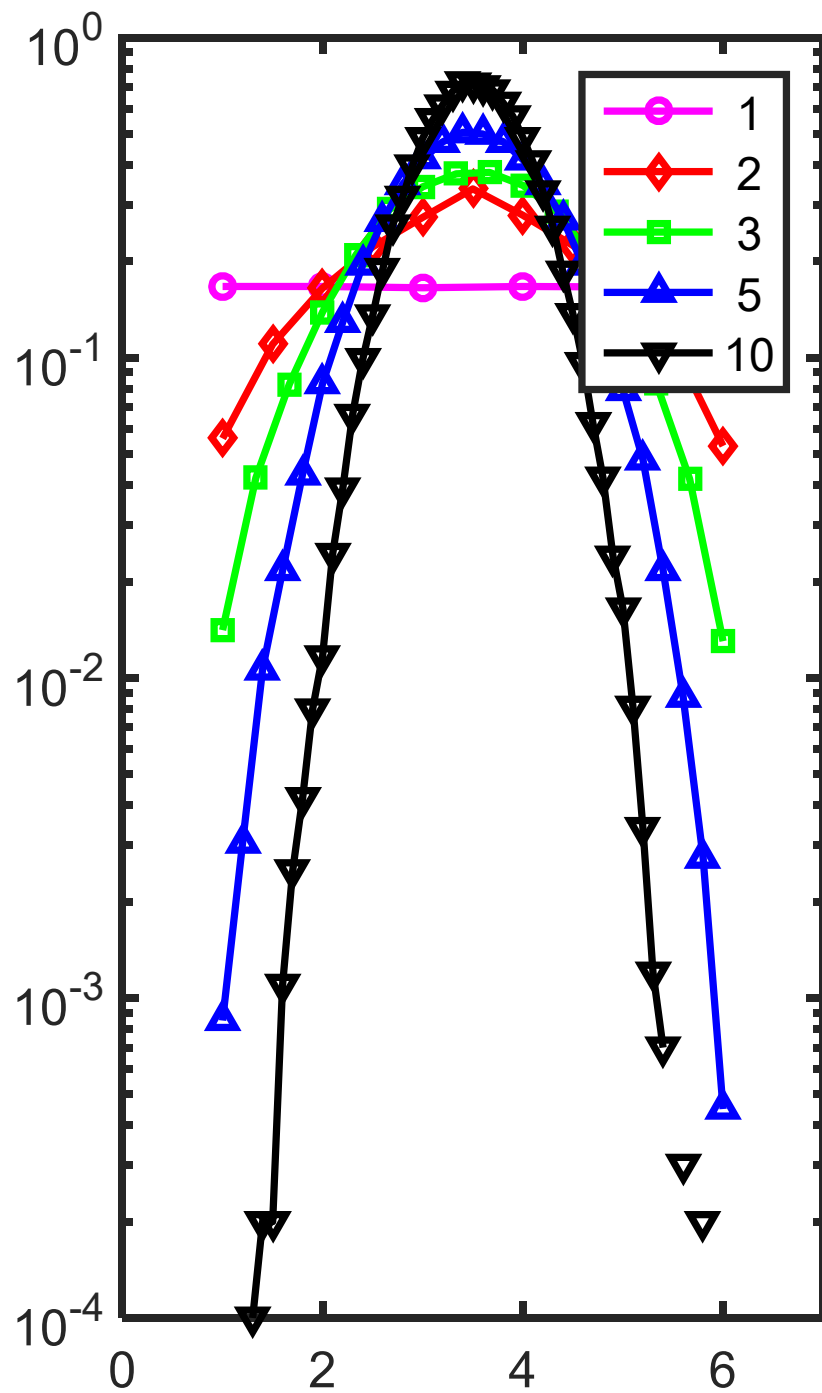
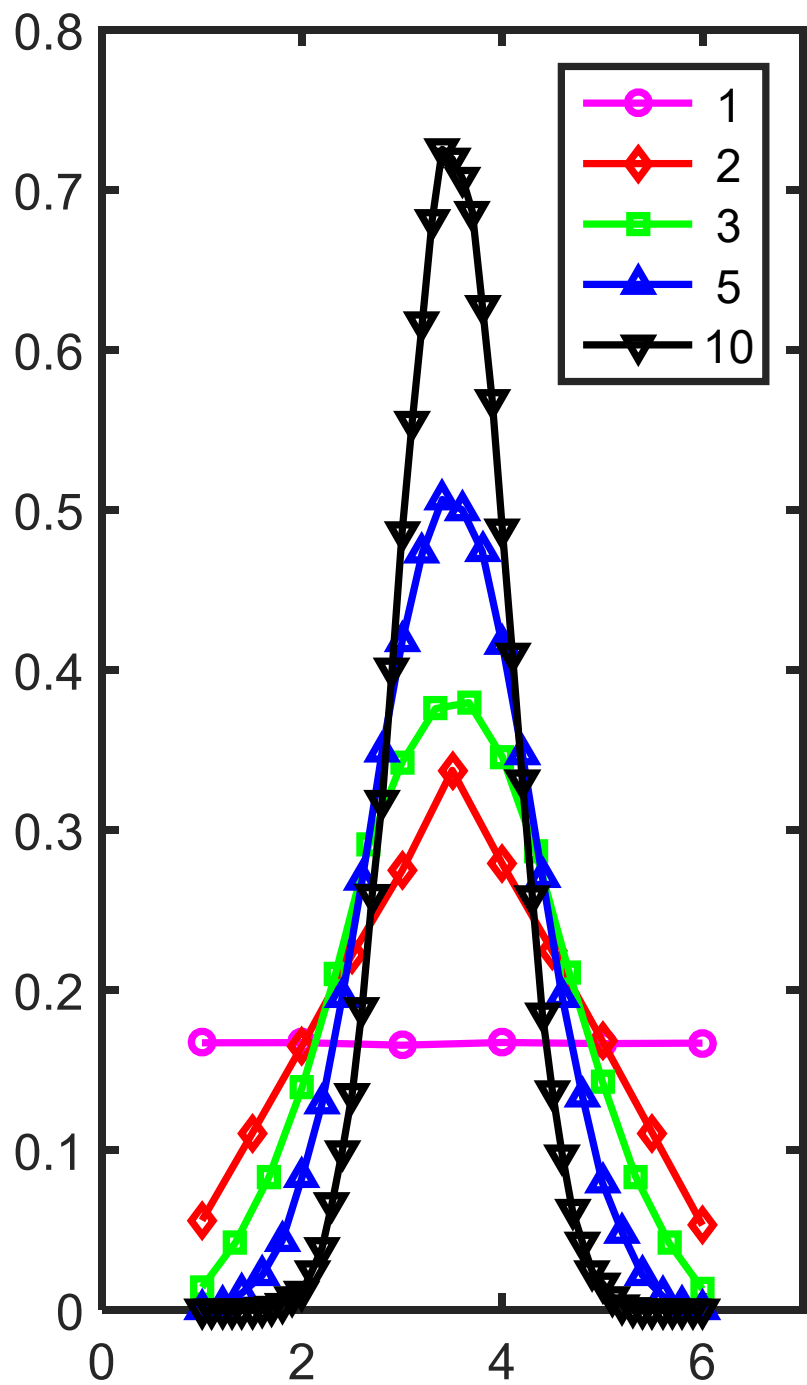
- Do a numerical experiment: generate a **sample of size n** by rolling **n fair dice**
- Calculate the **sample mean**
$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$$
- Repeat **Stats=100,000 times**
- Generate **PDFs of sample means** for different samples sizes: $n=1$, $n=2$, $n=3$, $n=5$, and $n=10$
- **Plot them in the same** (semi-logarithmic) **figure**
- **What do you see?**
- Template is at the website:
`central_limit_theorem_template.m`

How did I do it?

- **Stats=100000;**
- **figure;**
- **for n=[1,2,3,5,10];**
- **r_sample=floor(6.*rand(Stats,n))+1;**
- **sample_mean=sum(r_sample,2)./n;**
- **step=1./n;**
- **[a,b1]=hist(sample_mean,1:step:6);**
- **pdf_r1=a./sum(a)./step;**
- **semilogy(b1,pdf_r1,'o-'); hold on;**
- **end;**
- **legend('1','2','3','5','10');**

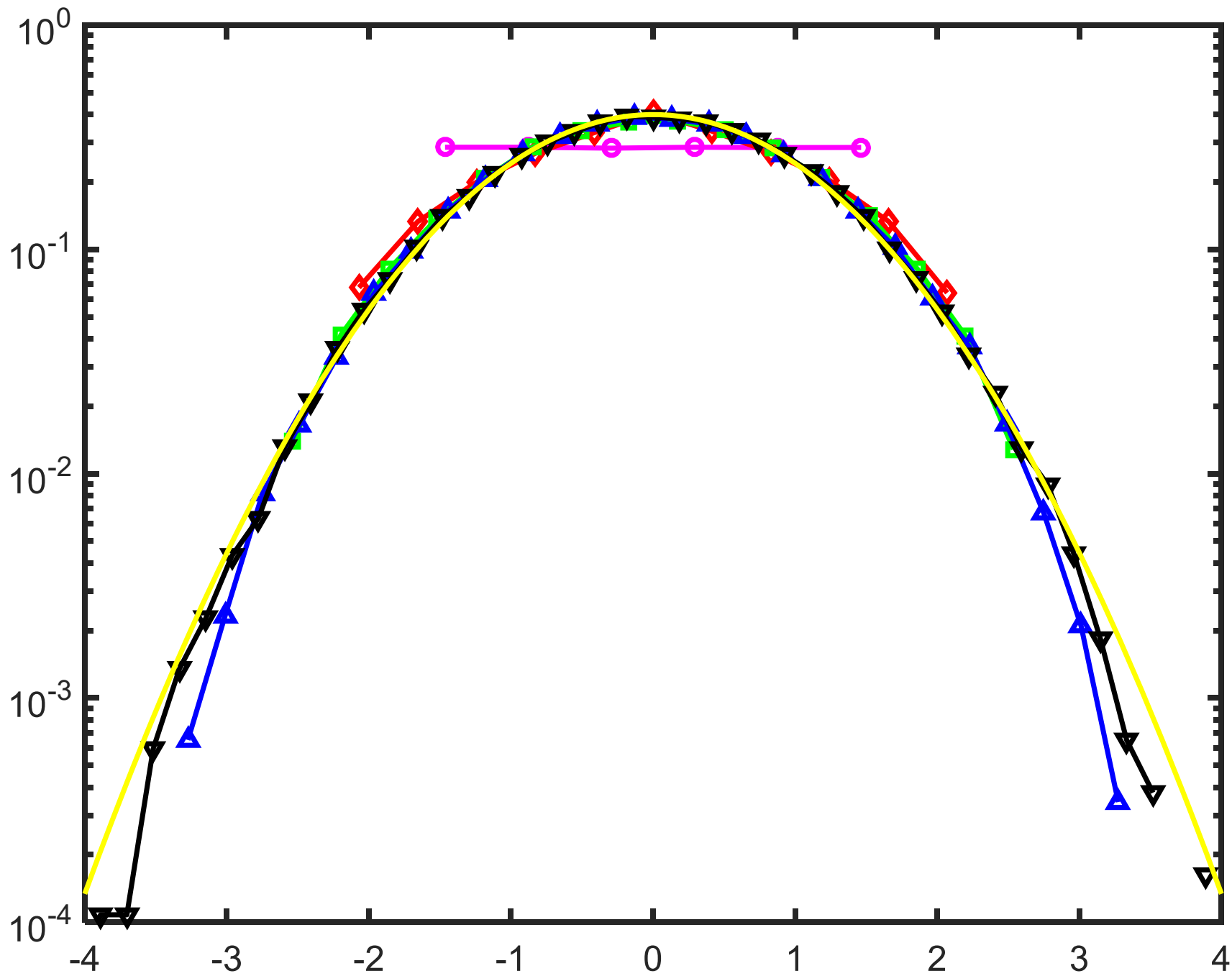
Matlab demonstration

- `Stats=100000; N=10;`
- `r_table=floor(6.*rand(Stats,N))+1;`
- `%%`
- `r1=r_table(:,1);`
- `step=1; [a,b1]=hist(r1,1:step:6);`
- `pdf_r1=a./sum(a)./step;`
- `figure; hold on; subplot(1,2,1); plot(b1,pdf_r1,'mo-'); hold on; axis([0 7 0 0.2]); subplot(1,2,2);`
- `semilogy(b1,pdf_r1,'mo-'); hold on; axis([0 7 1e-3 1]);`
- `%%`
- `r2=(r_table(:,1)+r_table(:,2))./2;`
- `step=0.5; [a,b2]=hist(r2,1:step:6); pdf_r2=a./sum(a)./step;`
- `subplot(1,2,1); plot(b2,pdf_r2,'rd-'); axis([0 7 0 0.4]); subplot(1,2,2); semilogy(b2,pdf_r2,'rd-');`
- `%%`
- `r3=(r_table(:,1)+r_table(:,2)+r_table(:,3))./3;`
- `step=1./3; [a,b3]=hist(r3,1:step:6); pdf_r3=a./sum(a)./step;`
- `subplot(1,2,1); plot(b3,pdf_r3,'gs-'); axis([0 7 0 0.4]); subplot(1,2,2); semilogy(b3,pdf_r3,'gs-');`
- `%%`
- `r5=sum(r_table(:,1:5),2)./5;`
- `step=1./5; [a,b5]=hist(r5,1:step:6); pdf_r5=a./sum(a)./step;`
- `subplot(1,2,1); plot(b5,pdf_r5,'b^-'); axis([0 7 0 0.6]); subplot(1,2,2); semilogy(b5,pdf_r5,'b^-'); axis([0 7 1e-4 1]);`
- `%%`
- `r10=sum(r_table(:,1:10),2)./10;`
- `step=1./10; [a,b10]=hist(r10,1:step:6); pdf_r10=a./sum(a)./step;`
- `subplot(1,2,1); plot(b10,pdf_r10,'kv-'); axis([0 7 0 0.8]); legend(num2str([1,2,3,5,10]'));`
- `subplot(1,2,2); semilogy(b10,pdf_r10,'kv-'); legend(num2str([1,2,3,5,10]'));`



Matlab demonstration; part 2

- `%%Now plot all of them normalized to 0 and std 1`
- `sigma=sqrt(35/12);`
- `mu=3.5;`
- `figure;`
- `sigma1=sigma;`
- `semilogy((b1-mu)./sigma1,pdf_r1.*sigma1,'mo-');`
- `axis([-4 4 1e-3 1]);`
- `hold on;`
- `%%`
- `sigma2=sigma./sqrt(2);`
- `semilogy((b2-mu)./sigma2,pdf_r2.*sigma2,'rd-');`
- `%%`
- `sigma3=sigma./sqrt(3);`
- `semilogy((b3-mu)./sigma3,pdf_r3.*sigma3,'gs-');`
- `%%`
- `sigma5=sigma./sqrt(5);`
- `semilogy((b5-mu)./sigma5,pdf_r5.*sigma5,'b^-');`
- `axis([-4 4 1e-4 1]);`
- `%%`
- `sigma10=sigma./sqrt(10);`
- `semilogy((b10-mu)./sigma10,pdf_r10.*sigma10,'kv-');`
- `axis([-4 4 1e-4 1]);`
- `%%`
- `%Let's see how well does the Gaussian fits it`
- `x=-4:0.1:4;`
- `semilogy(x,1./sqrt(2*pi)*exp(-x.^2./2),'y-');`



Credit: XKCD
comics

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WHY IS PROGRAMMING SO HARD
WHY IS THERE A 0 OHM RESISTOR
WHY DO AMERICANS HATE SOCCER
WHY DO RHYMES SOUND GOOD
WHY DO TREES DIE
WHY IS THERE NO SOUND ON CNN
WHY AREN'T POKEMON REAL
WHY AREN'T BULLETS SHARP
WHY DO DREAMS SEEM SO REAL

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Example 7-1: Resistors

An electronics company manufactures resistors having a mean resistance of 100 ohms and a standard deviation of 10 ohms. What is the approximate probability that a random sample of $n = 25$ resistors will have an average resistance of less than 95 ohms?

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$$\mu = 100 \text{ ohms}, \quad \sigma = 10 \text{ ohms}, \quad n = 25$$

$$\mu_{\bar{x}} = \mu; \quad \sigma_{\bar{x}_n} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = \frac{10}{5} = 2 \text{ ohms}$$

$$Z_{\bar{x}} = \frac{95 - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{95 - 100}{2} = -2.5$$

$$\begin{aligned} \text{Prob}(\bar{X} < 95) &= \Phi(Z_{\bar{x}}) = \Phi(-2.5) = \\ &= 0.0062 \end{aligned}$$

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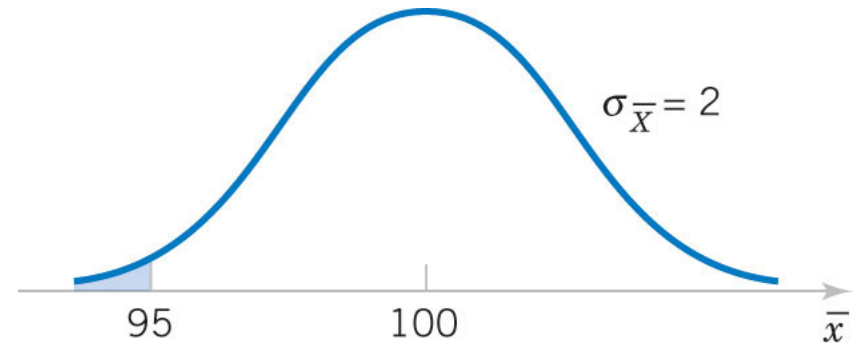


Figure 7-2 Desired probability is shaded

Answer:

$$\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2.0$$
$$\Phi\left(\frac{\bar{X} - \mu}{\sigma_{\bar{X}}}\right) = \Phi\left(\frac{95 - 100}{2}\right)$$
$$= \Phi(-2.5) = 0.0062$$

Two Populations

We have two independent populations. What is the distribution of the difference of their sample means?

The sampling distribution of $\bar{X}_1 - \bar{X}_2$ has the following mean and variance:

$$\mu_{\bar{X}_1 - \bar{X}_2} = \mu_{\bar{X}_1} - \mu_{\bar{X}_2} = \mu_1 - \mu_2$$

$$\sigma_{\bar{X}_1 - \bar{X}_2}^2 = \sigma_{\bar{X}_1}^2 + \sigma_{\bar{X}_2}^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Sampling Distribution of a Difference in Sample Means

- **If** we have two independent populations with means μ_1 and μ_2 , and variances σ_1^2 and σ_2^2 ,
- **And if** \bar{X}_1 and \bar{X}_2 are the sample means of two independent random samples of sizes n_1 and n_2 from these populations:
- **Then** the sampling distribution of:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (7-4)$$

is approximately standard normal, if the conditions of the central limit theorem apply.

- **If** the two populations are normal, **then** the sampling distribution is exactly standard normal.

Example 7-3: Aircraft Engine Life

The effective life of a component used in jet-turbine aircraft engines is a random variable with $\mu_{\text{old}}=5000$ hours and $\sigma_{\text{old}}=40$ hours (old). The engine manufacturer introduces an improvement into the manufacturing process for this component that changes the parameters to $\mu_{\text{new}}=5050$ hours and $\sigma_{\text{new}}=30$ hours (new).

Random samples of 16 components manufactured using “old” process and 25 components using “new” process are chosen.

What is the probability new sample mean is at least 25 hours longer than old?

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$$\sigma_{\bar{X}_{old}} = \frac{\sigma_{old}}{\sqrt{16}} = 10 \text{ hrs}$$

$$\sigma_{\bar{X}_{new}} = \frac{\sigma_{new}}{\sqrt{25}} = 6 \text{ hrs}$$

$$\sigma_{TOT} = \sqrt{\sigma_{\bar{X}_{old}}^2 + \sigma_{\bar{X}_{new}}^2} = \sqrt{100 + 36} \approx 11.7 \text{ hrs}$$

$$\mu_{new} - \mu_{old} = 50 \text{ hrs}$$

$$z = \frac{25 - (50)}{11.7} = -2.14$$
$$\text{Prob}(z > -2.14) = 0.9840$$

Example 7-3: Aircraft Engine Life

The effective life of a component used in jet-turbine aircraft engines is a normal-distributed random variable with parameters shown (old). The engine manufacturer introduces an improvement into the manufacturing process for this component that changes the parameters μ and σ as shown (new).

Random samples are selected from the “old” process and “new” process as shown.

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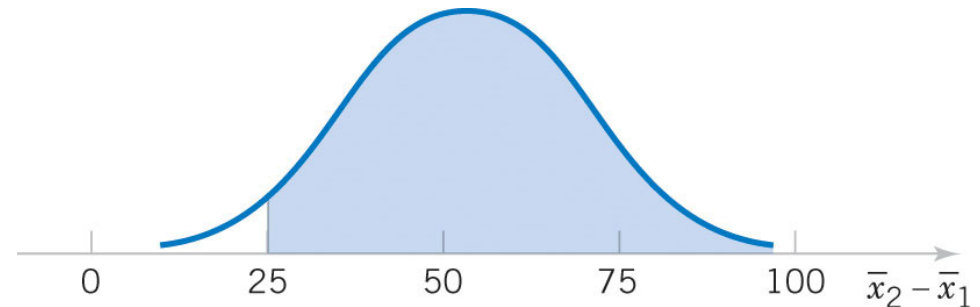


Figure 7-4 Sampling distribution of the sample mean difference.

	Process		
	Old (1)	New (2)	Diff (2-1)
$\mu =$	5,000	5,050	50
$\sigma =$	40	30	50
$n =$	16	25	
Calculations			
$s / \sqrt{n} =$	10	6	11.7
		$z =$	-2.14
	$P(\bar{x}_2 - \bar{x}_1 > 25) = P(Z > z) =$		0.9840

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Descriptive statistics:

Point estimation:

Point Estimation

- A sample was collected: X_1, X_2, \dots, X_n
- We suspect that sample was drawn from a random variable distribution $f(x)$
- $f(x)$ has k parameters that we do not know
- Point estimates are estimates of the parameters of the $f(x)$ describing the population based on the sample
 - For exponential PDF: $f(x) = \lambda \exp(-\lambda x)$ one wants to estimate λ
 - For Bernoulli PDF: $p^x(1-p)^{1-x}$ one wants to estimate p
 - For normal PDF one wants to estimate both μ and σ
- Point estimates are uncertain: therefore, we can talk of averages and standard deviations of point estimators

Point Estimator

A **point estimate** of some parameter θ describing population random variable is a single numerical value $\hat{\theta}$ depending on all values x_1, x_2, \dots, x_n in the sample.

The sample statistic (whis a random variable $\hat{\Theta}$ defined by a function $\hat{\Theta}(X_1, X_2, \dots, X_n)$) is called the **point estimator**.

- There could be **multiple choices** for the point estimator of a parameter.
- To estimate the **mean of a population**, we could choose the:
 - **Sample mean**
 - Sample median
 - Peak of the histogram
 - $\frac{1}{2}$ of (largest + smallest) observations of the sample.
- We need to develop criteria to compare estimates using statistical properties.

Unbiased Estimators Defined

The point estimator $\hat{\Theta}$ is an **unbiased estimator**

for the parameter θ if:

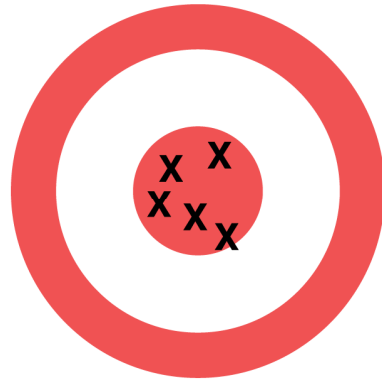
$$E(\hat{\Theta}) = \theta \quad (7-5)$$

If the estimator is not unbiased, then the difference:

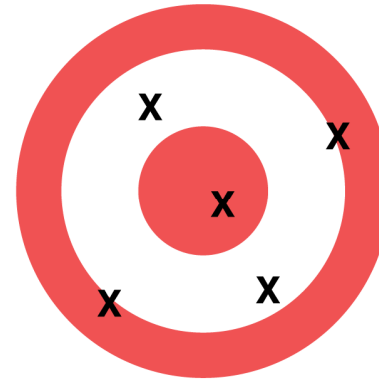
$$E(\hat{\Theta}) - \theta \quad (7-6)$$

is called the **bias** of the estimator $\hat{\Theta}$.

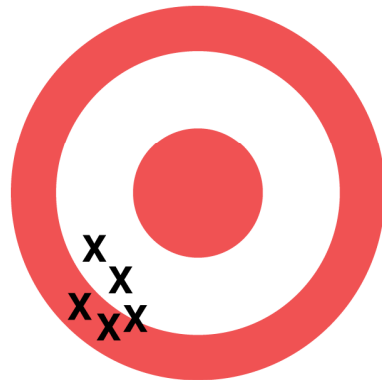
Bias vs Noise



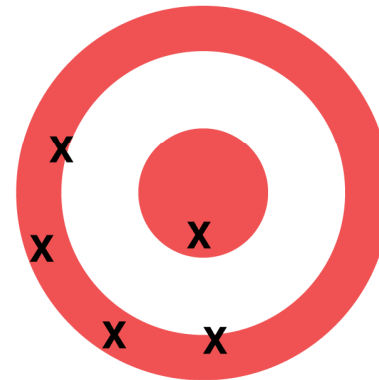
ACCURATE



NOISY



BIASED



BIASED & NOISY

Mean Squared Error

The **mean squared error** of an estimator $\hat{\Theta}$ of the parameter θ is defined as:

$$\text{MSE}(\hat{\Theta}) = E(\hat{\Theta} - \theta)^2 \quad (7-7)$$

Can be rewritten as

$$\begin{aligned} &= E[\hat{\Theta} - E(\hat{\Theta})]^2 + [\theta - E(\hat{\Theta})]^2 \\ &= V(\hat{\Theta}) + (\text{bias})^2 \end{aligned}$$

Statistic #1: Sample Mean

If the values of n observations in a random sample are denoted by x_1, x_2, \dots, x_n , the **sample mean** is

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum_{i=1}^n x_i}{n} \quad (6-1)$$

New random variable \bar{X} is a linear combination of n independent identically distributed variables X_1, X_2, \dots, X_n

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

Sample mean \bar{x} is drawn from a random variable

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$E(\bar{X}) = \frac{n \cdot E(X_i)}{n} = \frac{n \cdot \mu}{n} = \mu$$

Sample mean, \bar{X} , is an unbiased estimator of the population mean, μ

Sample variance S^2 –
is an estimator of
the population variance σ^2

Sample Variance

If n observations in a sample are denoted by x_1, x_2, \dots, x_n , the **sample variance** is

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1} \quad (6-3)$$

If one knows the **population average**, μ , one **divides by n** to estimate the variance

$$s(\mu)^2 = \frac{\sum_{i=1}^n (x_i - \mu)^2}{n}$$

Why divide by n-1 instead of n?

- The **sample mean \bar{x} is on average closer** to points x_1, x_2, \dots, x_n than **the true mean μ**
$$\sum_{i=1}^n (x_i - \bar{x})^2 \geq \sum_{i=1}^n (x_i - \mu)^2$$
- Consider a sample of size $n=1$.
Then $\bar{x} = x_1$ while $\mu \neq x_1$. Dividing by n gives $s^2 = 0$, while dividing by $n-1$ leaves **s^2 undefined (0/0)**
- For $n=2$, \bar{x} is exactly halfway between x_1 and x_2 making its **sum of squares smaller than** that of μ
- Dividing by $n-1$ on average corrects for a smaller sum of squares: **S^2 is an unbiased estimator of σ^2**

Show that s^2 is unbiased estimate of σ^2

$$\begin{aligned} E(s^2) &= E\left(\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}\right) = \frac{1}{n-1} E\left[\sum_{i=1}^n (X_i^2 + \bar{X}^2 - 2\bar{X}X_i)\right] \\ &= \frac{1}{n-1} E\left[\sum_{i=1}^n X_i^2 + n\bar{X}^2 - 2\bar{X}n\bar{X}\right] = \\ &= \frac{1}{n-1} E\left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right) = \frac{1}{n-1} (nE(X_i^2) - nE(\bar{X}^2)) \\ &= \frac{1}{n-1} \left(n(\mu^2 + \sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right) = \frac{n-1}{n-1} \sigma^2 = \underline{\underline{\sigma^2}} \end{aligned}$$

Example 7-4: Sample Variance S^2 is Unbiased

$$\begin{aligned} E(S^2) &= E\left(\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}\right) \\ &= \frac{1}{n-1} E\left[\sum_{i=1}^n (X_i^2 + \bar{X}^2 - 2\bar{X}X_i)\right] \\ &= \frac{1}{n-1} \left[E\left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right) \right] \\ &= \frac{1}{n-1} \left[\sum_{i=1}^n (\mu^2 + \sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right) \right] \\ &= \frac{1}{n-1} [n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2] = \frac{1}{n-1} [(n-1)\sigma^2] \end{aligned}$$

Credit: XKCD
comics

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WHY IS EARTH TILTED
WHY IS SPACE BLACK
WHY IS OUTER SPACE SO COLD
WHY ARE THERE PYRAMIDS ON THE MOON
WHY IS NASA SHUTTING DOWN



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Methods of Point Estimation

- We will cover two popular methodologies to create point estimates of a population parameter.
 - Method of moments
 - Method of maximum likelihood
- Each approach can be used to create estimators with varying degrees of biasedness and relative MSE efficiencies.

Method of moments for point estimation

What are moments?

- The p-th **population moment** of a random variable is the expected value of X^p
 - First moment: $\mu = \int_{-\infty}^{+\infty} x f(x) dx$
 - Second moment: $\mu^2 + \sigma^2 = \int_{-\infty}^{+\infty} x^2 f(x) dx$
 - p-th moment: $\int_{-\infty}^{+\infty} x^p f(x) dx$
 - The **population moment** relates to the entire population
- A **sample moment** is calculated like its population moments but for a finite sample
 - Sample first moment = sample mean = $\frac{1}{n} \sum_{i=1}^n x_i$
 - Sample p-th moment $\frac{1}{n} \sum_{i=1}^n x_i^p$

Moment Estimators

Let X_1, X_2, \dots, X_n be a random sample from either a probability mass function or a probability density function with p unknown parameters $\theta_1, \theta_2, \dots, \theta_p$.

The **moment estimators** $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_p$ are found by equating the first p population moments to the first p sample moments and solving the resulting simultaneous equations for the unknown parameters.

Exponential Distribution: Moment Estimator-1st moment

- Suppose that x_1, x_2, \dots, x_n is a random sample from an exponential distribution $f(x) = \lambda \exp(-\lambda x)$ with parameter λ .
- There is only one parameter to estimate, so equating population and sample first moments, we have one equation: $E(X) = \bar{x}$.
- $E(X) = 1/\lambda$ thus $\lambda = 1/\bar{x}$ is the 1st moment estimator.

Matlab exercise

- Generate 100,000 exponentially distributed random numbers with $\lambda=3$: $f(x)=\lambda\exp(-\lambda x)$
 - Use `random('Exponential'...)` but read the manual to know how to introduce parameters.
- Get a moment estimate of lambda based on the 1st moment
- Get a moment estimate of lambda based on the 2nd moment
 - Second moment of the exponential distribution is $E(X^2) = E(X)^2 + \text{Var}(X) = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2$
- Get a moment estimate of lambda based on the 20th moment
 - Generally, p-th moment of the exponential distribution is $E(X^p) = p!/\lambda^p$

How I solved it

- Stats=100000;
- Y=random('Exponential', 1/3, Stats, 1);
%parametrization in MATLAB is 1/lambda
- 1/mean(Y) %matching the first moment
% ans = 3.0086
- sqrt(2/mean(Y.^2)) %matching the second moment
% ans = 3.0081
- (factorial(20)/mean(Y.^20))^(1./20) %matching the 20th moment

Credit: XKCD
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