University of Illinois **Spring 2024**

ECE 563: Problem Set 2
Rate Distortion and Coded Side Information

Released: Monday, January 29
Due: Monday, February 12

1. **[Inequalities]**

   (a) State and prove Fano's inequality.

   (b) (Optional) Prove Shannon's inequality, which is (rarely) used in similar applications:

   $$P_e \geq \tfrac{1}{6} \frac{H(X|Y)}{\log M + \log\log M - \log H(X|Y)}$$

2. **[Erokhin]**
   Consider a source random variable X with the Hamming distortion measure.

   (a) Prove that
   $$R(D) \geq H(X) - D\log(|\mathcal{X} - 1) - h_2(D)$$
   for $0 \leq D \leq D_{\max}$.

   (b) Show that this lower bound (which is a special case of the Shannon lower bound) is tight if X is uniformly distributed on $\mathcal{X}$.

3. **[Erase]**
   Consider $X \sim Bernoulli(1/2)$ and let the distortion measure be given by

   $$d(x, \hat{x}) = \begin{bmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{bmatrix}$$

   Calculate the rate-distortion function for this source and suggest a simple scheme to achieve any value on the rate-distortion curve.

4. **[Gu et al.]**
   WeiHsin Gu, Ralf Koetter, Michelle Effros, and Tracey Ho wrote a paper, "On Source Coding with Coded Side Information for a Binary Source with Binary Side Information," in ISIT 2007 that explicitly found the rate region for a specific example of the source coding with side information problem.

   (a) Read through their paper and describe the intuitions you can get about the nature of the optimal auxilliary random variable. This should be some sentences in words.

   (b) (Optional) See if you can develop any other closed-form examples using their techniques.

5. **[Wolf's Problem]**
   (Optional) Jack Wolf, in a 2004 paper at CISS entitled "Source Coding for a Noiseless Broadcast Channel," revisited the original proof of the Slepian-Wolf problem that was done using error-correcting codes rather than random binning. He basically argued as follows. Alice possesses a file modeled as a random binary string A_1^N and Bob possesses a corrupted version of the file, B_1^N, modeled as passing A_1^N through a BSC with crossover probability p. Suppose Alice knows Bob's corrupted version, and she wishes to inform Bob of A_1^N using a short message. Alice could simply compute the difference sequence $A_1^N \oplus B_1^N$ (where $\oplus$ denotes addition modulo 2), compress it losslessly, and send this to Bob, requiring $Nh_2(p)$ bits. Alternatively, Alice could ignore her knowledge of B_1^N, and simply perform Slepian-Wolf binning of A_1^N and send the bin index. This also requires $Nh_2(p)$ bits in the large-N limit. While

both strategies are equally efficient from a rate perspective, the latter will require a significantly higher decoding complexity.

Wolf then went on to add a third user, Charles, who also has an independently corrupted version C_1^N of the original string A_1^N. Again, it is assumed that Alice knows the sequence C_1^N (as well as B_1^N), and the problem is now for Alice to inform both Bob and Charles of A_1^N in an efficient way using a single broadcast that everyone can hear noiselessly. She could send Bob's error sequence $A_1^N \oplus B_1^N$ over the broadcast channel followed by Charles' error sequence $A_1^N \oplus C_1^N$ using a total of $2Nh_2(p)$ bits. Or as before, she could broadcast the Slepian-Wolf bin index of A_1^N, requiring $Nh_2(p)$ bits. The latter scheme is now twice as efficient as the former, but still requires a significant increase in decoding complexity. The question posed by Wolf is: Is there a third strategy that has the efficiency of binning, but the "coding complexity" of merely sending the difference sequence? See if you can come up with such a scheme. (Note that we don't know how to do this.)