Spatially Coupled LDPC Codes Constructed From Protographs

Mitchell, D. G., Lentmaier, M., & Costello, D. J. (2015). Spatially coupled LDPC codes constructed from protographs. *IEEE Transactions on Information Theory*, *61*(9), 4866-4889.

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Outline

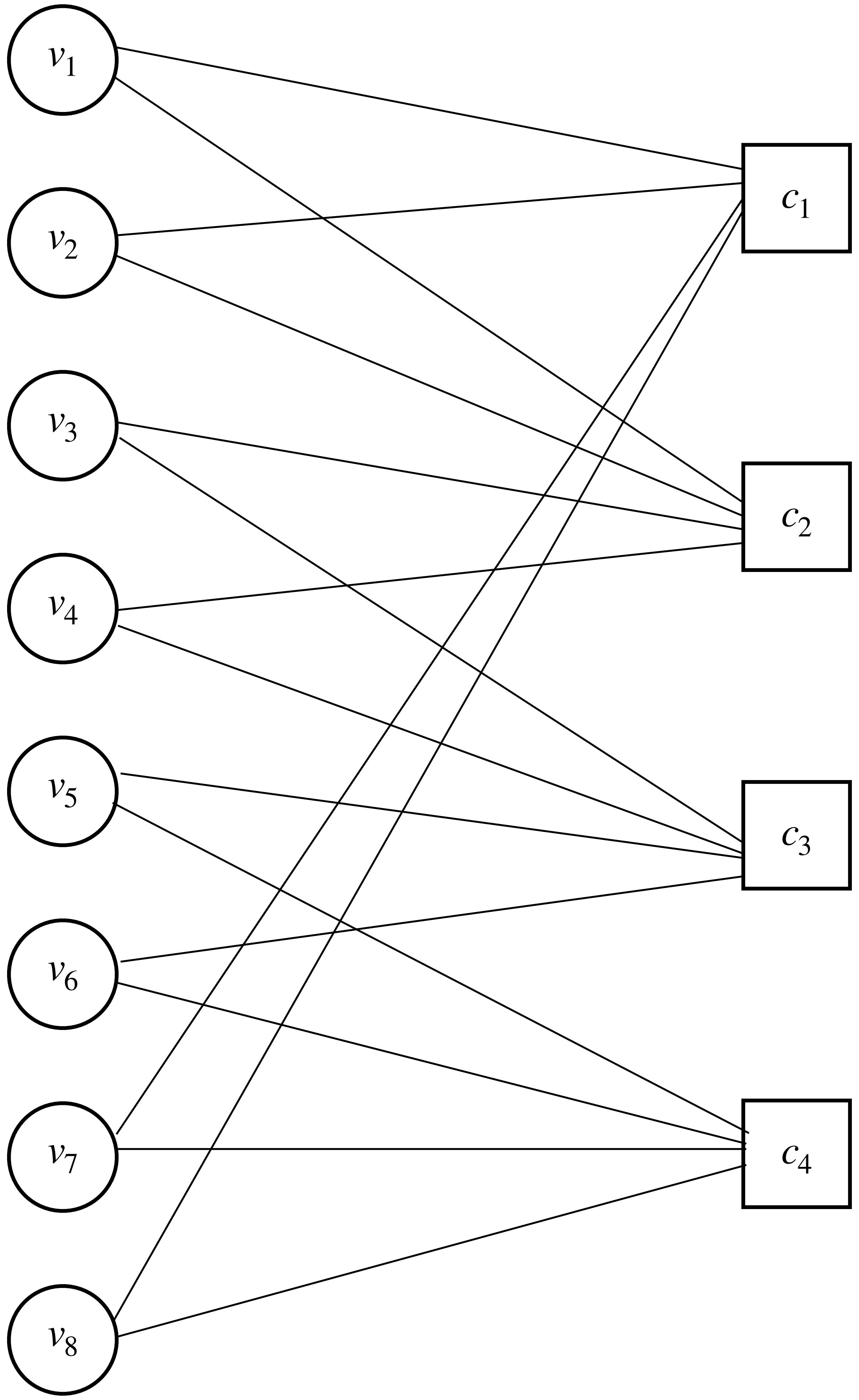
- Preliminaries
- BP threshold of SC-LDPC
- Minimum distance of SC-LDPC
- Numerical results

Preliminaries

- Regular/irregular LDPC codes
- Message passing and BP threshold
- Protograph, lifting and edge spreading

LDPC code—Tanner graph

Bipartite graph represents parity-check equations



(j, k)-regular LDPC code
Each variable node has the same degree, and each check nodes has the same degree

j: degree of variable nodes

k: degree of check nodes

<u>Example</u>

(2,4)-LDPC code

$$c_1 : v_1 + v_2 + v_7 + v_8 = 0$$

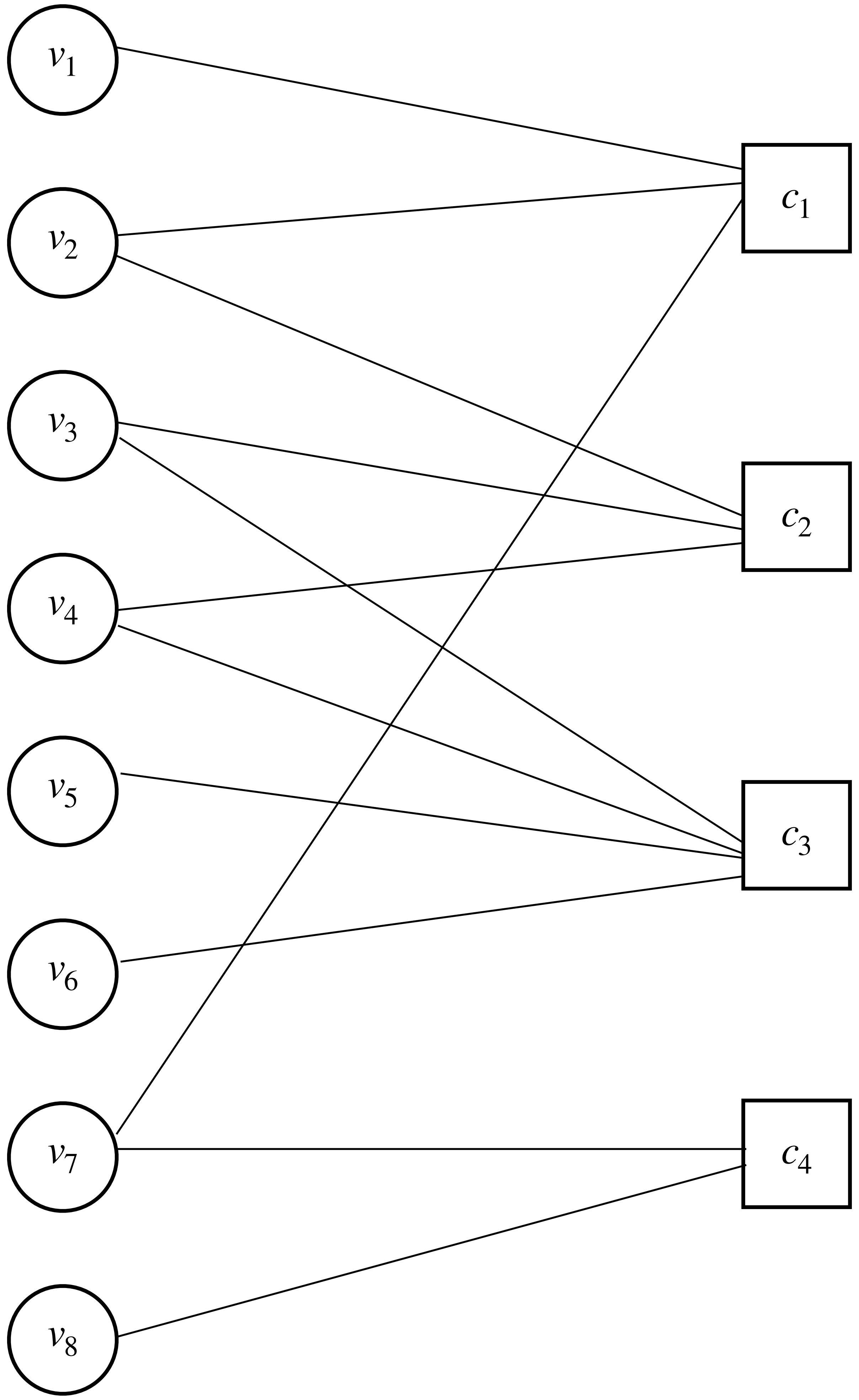
$$c_2 : v_1 + v_2 + v_3 + v_4 = 0$$

$$c_3 : v_3 + v_4 + v_5 + v_6 = 0$$

$$c_4 : v_5 + v_6 + v_7 + v_8 = 0$$

LDPC code—Tanner graph

Bipartite graph represents parity-check equations



irregular LDPC code

Defined by degree distribution functions

$$\lambda(x) = \sum_{i=1}^{d_v} \lambda_i x^{i-1} \quad \rho(x) = \sum_{i=2}^{d_c} \rho_i x^{i-1}$$

$d_v(d_c)$: max degree of variable(check) nodes

$\lambda_i(\rho_i)$: fraction of edges connected to variable(check) nodes of degree i

Example

$$\lambda(x) = \frac{1}{3} + \frac{2}{3}x \quad \rho(x) = \frac{1}{6}x + \frac{1}{2}x^2 + \frac{1}{3}x^3$$

LDPC code—Design rate

Given a Tanner graph of the LDPC code, the design rate is defined as

$$R = 1 - \frac{\#(\text{check nodes})}{\#(\text{variable nodes})}$$

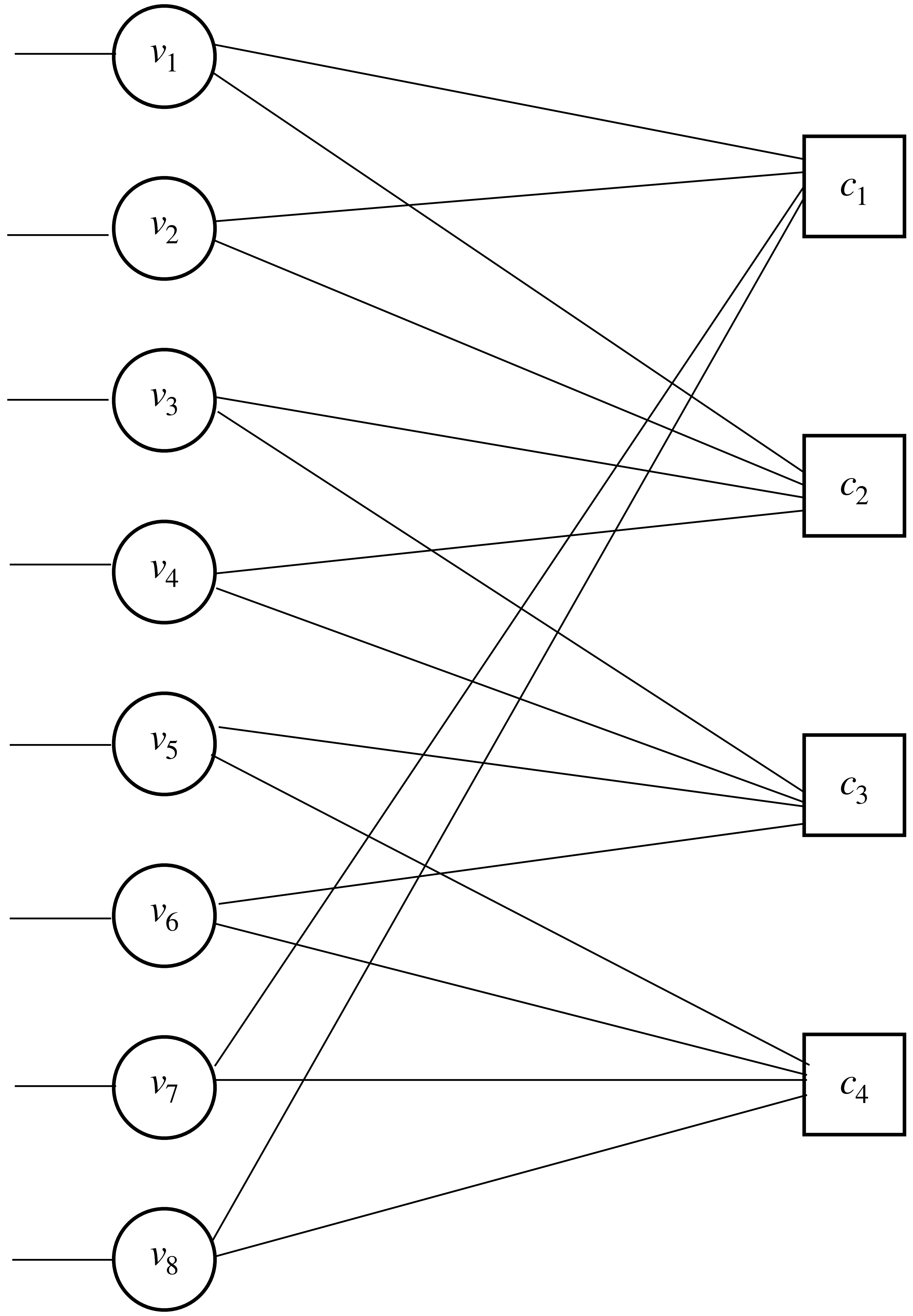
For a (j, k)-regular LDPC code, $R = 1 - \frac{j}{k}$

For an irregular LDPC code with degree distribution (λ, ρ),

$$R = 1 - \frac{\sum_j \frac{\rho_j}{j}}{\sum_k \frac{\rho_k}{k}} = 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx}$$

LDPC decoding—message passing

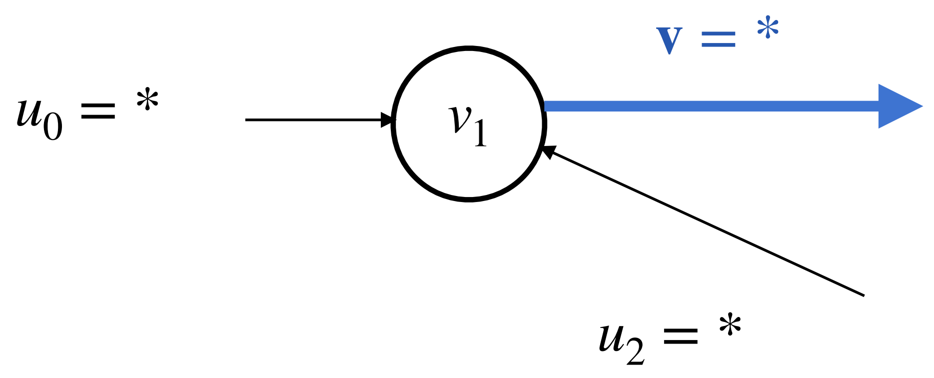
BEC as example



Variable to check message

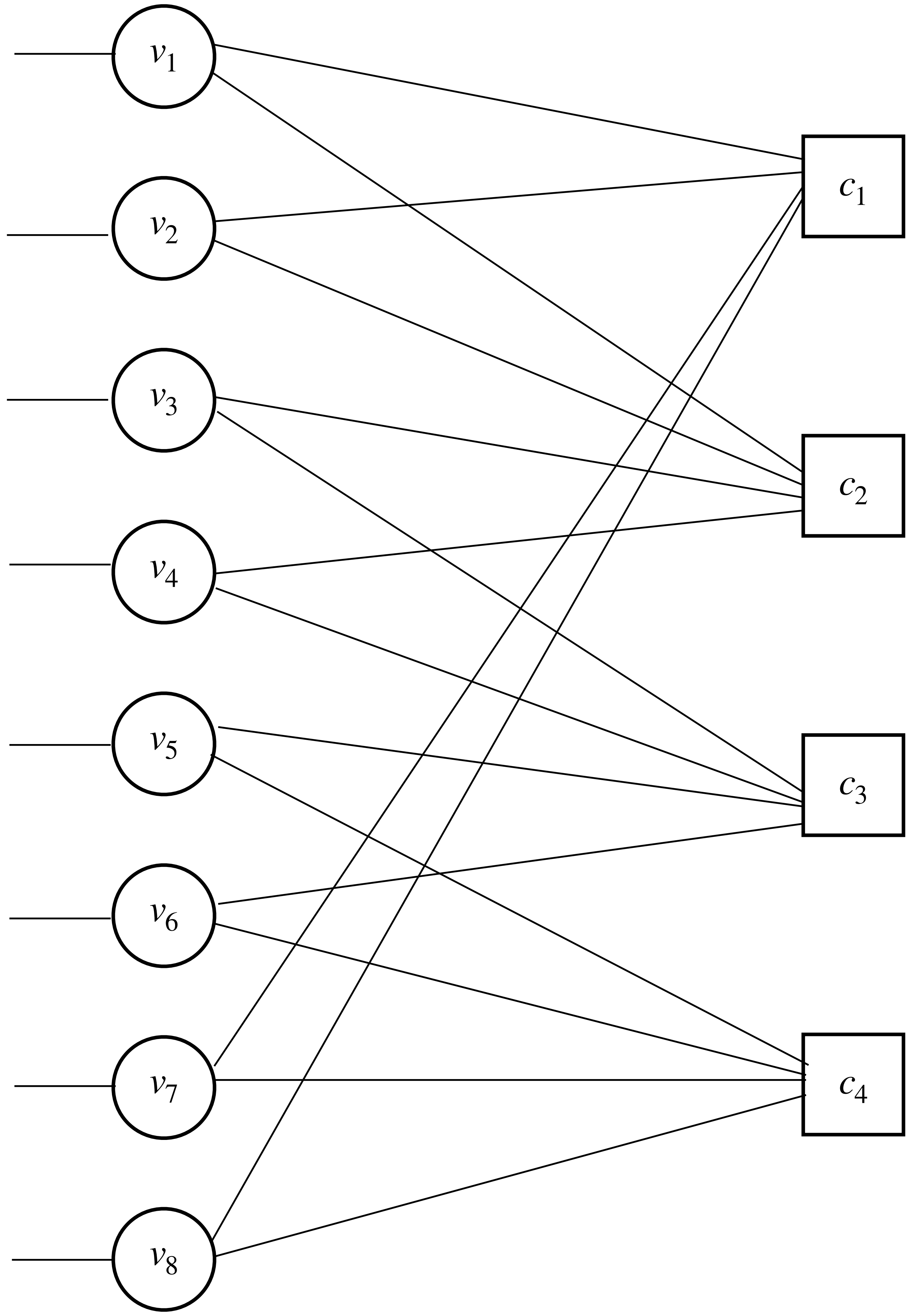
Aggregate information from all *other* check nodes

- All other incoming messages are *, send *



LDPC decoding—message passing

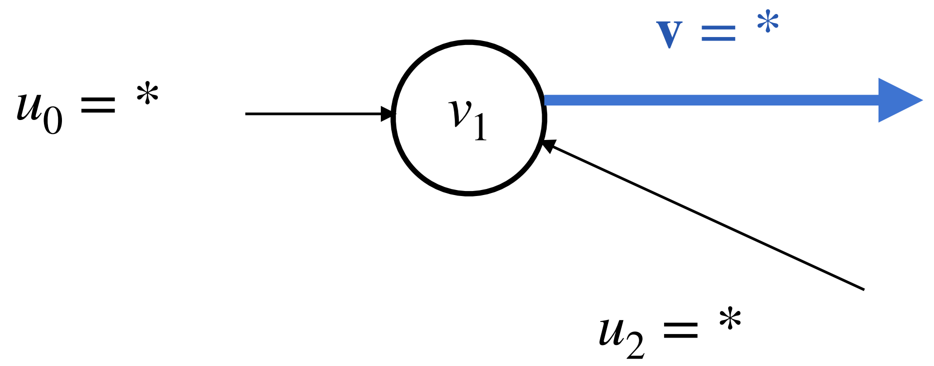
BEC as example



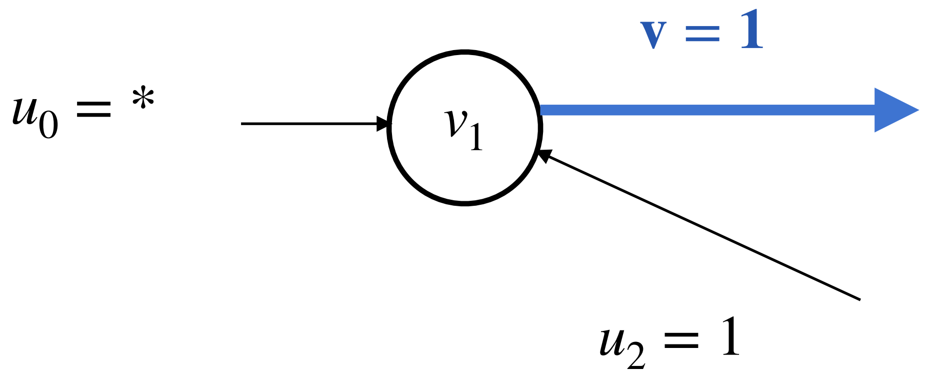
Variable to check message

Aggregate information from all *other* check nodes

- All other incoming messages are $*$, send $*$



- Any other incoming messages is $b \in \{0,1\}$, send b. If $u_0 = *$, set $u_0 = b$.



LDPC decoding—message passing

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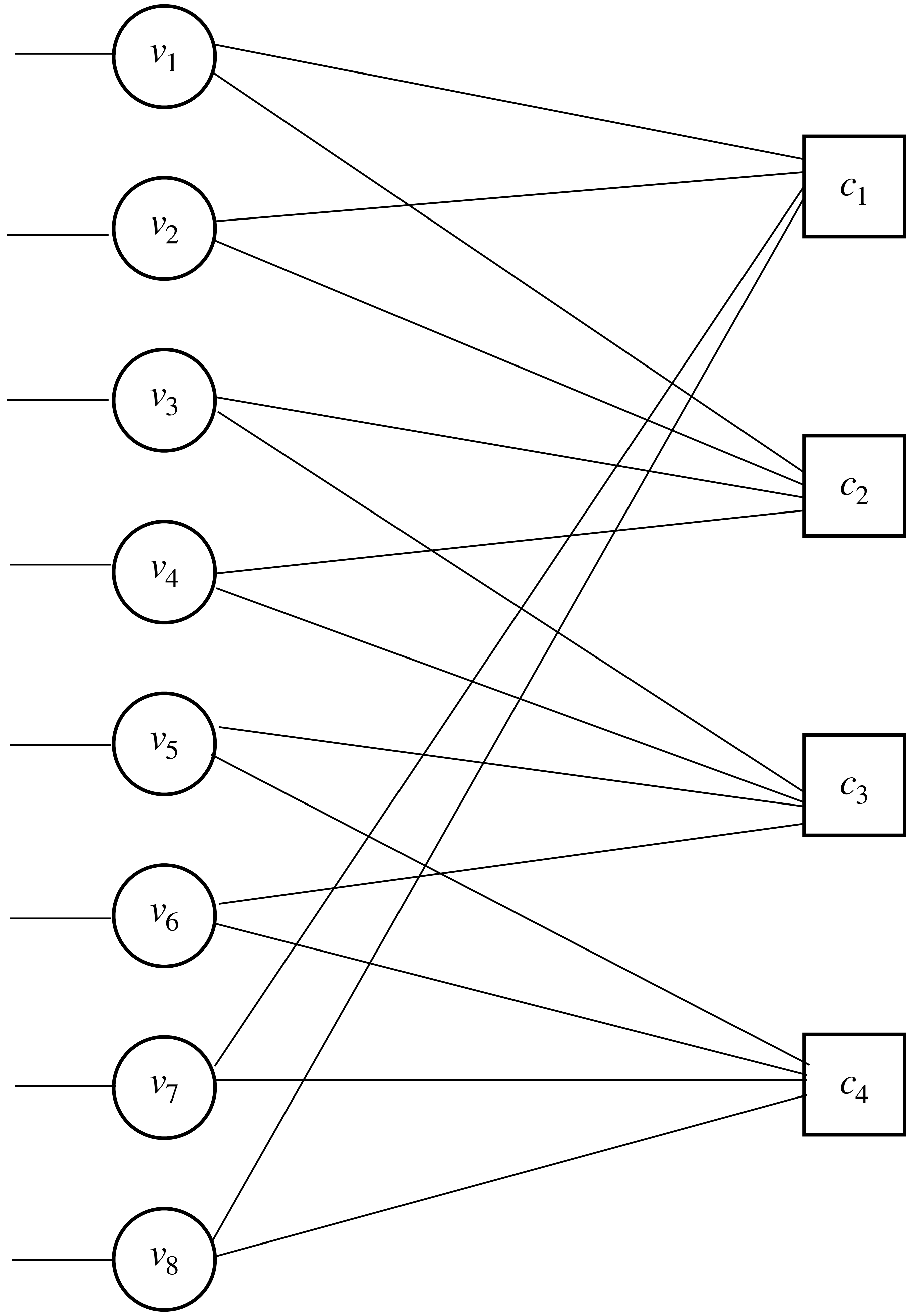

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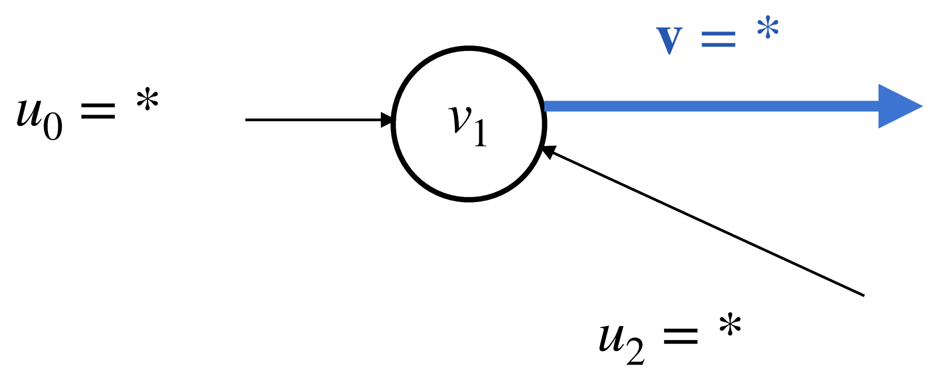

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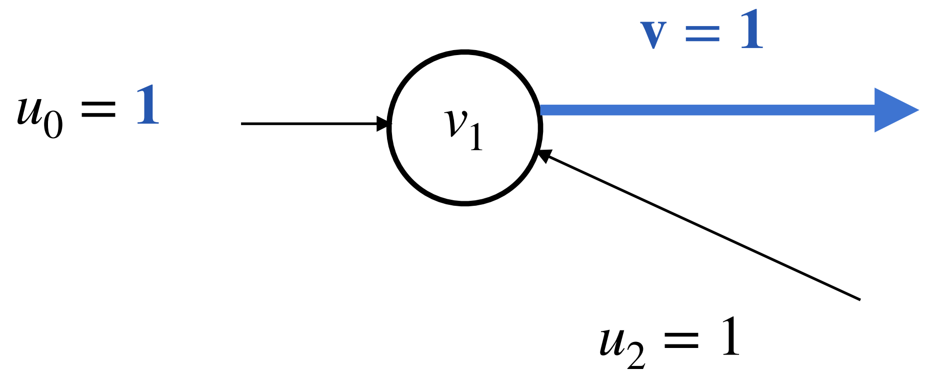
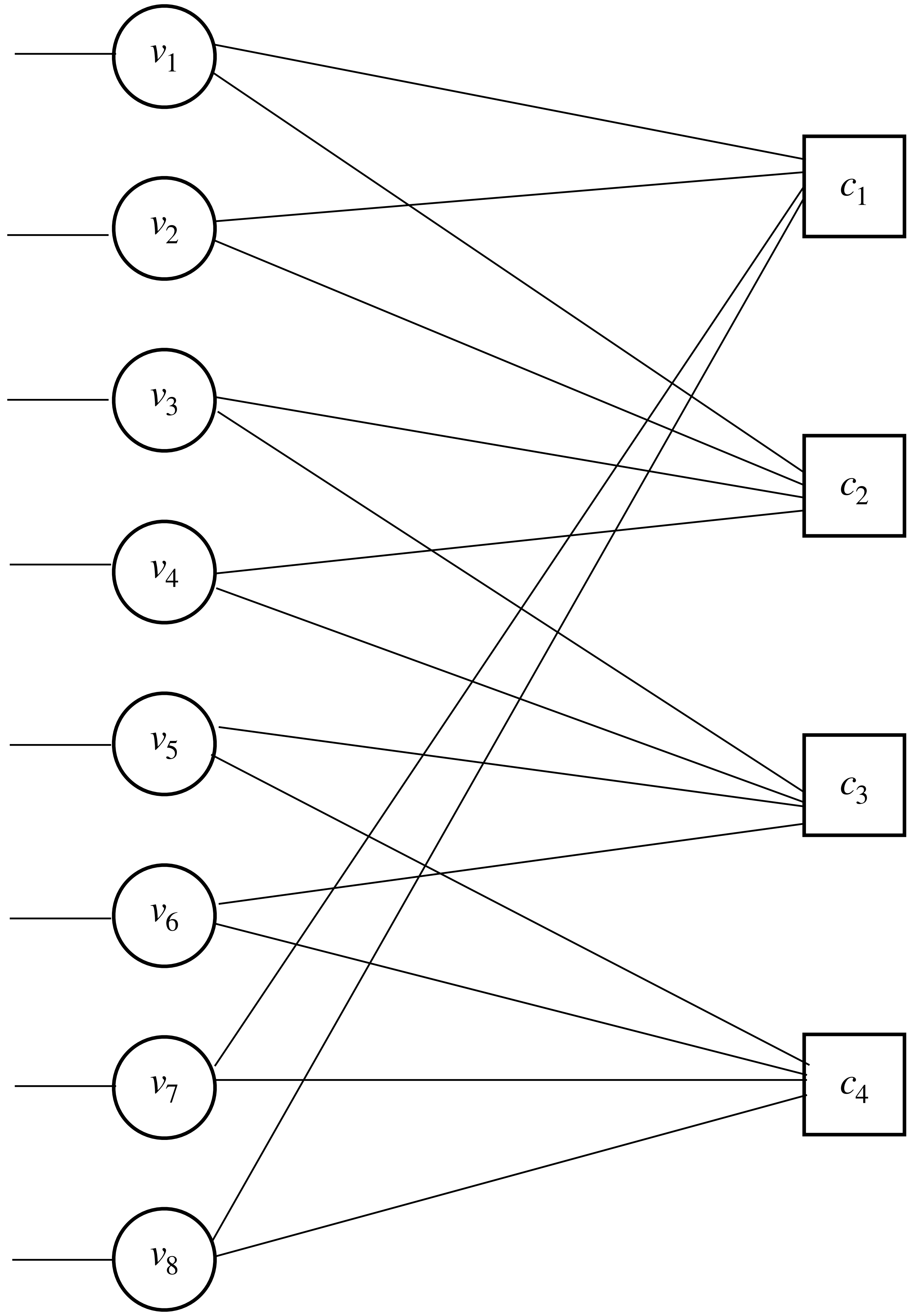


LDPC decoding—message passing

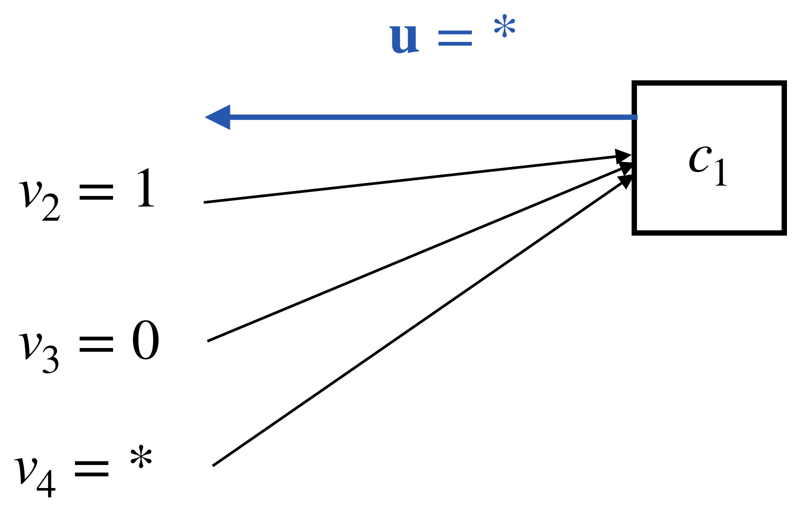
BEC as example



Check to variable message

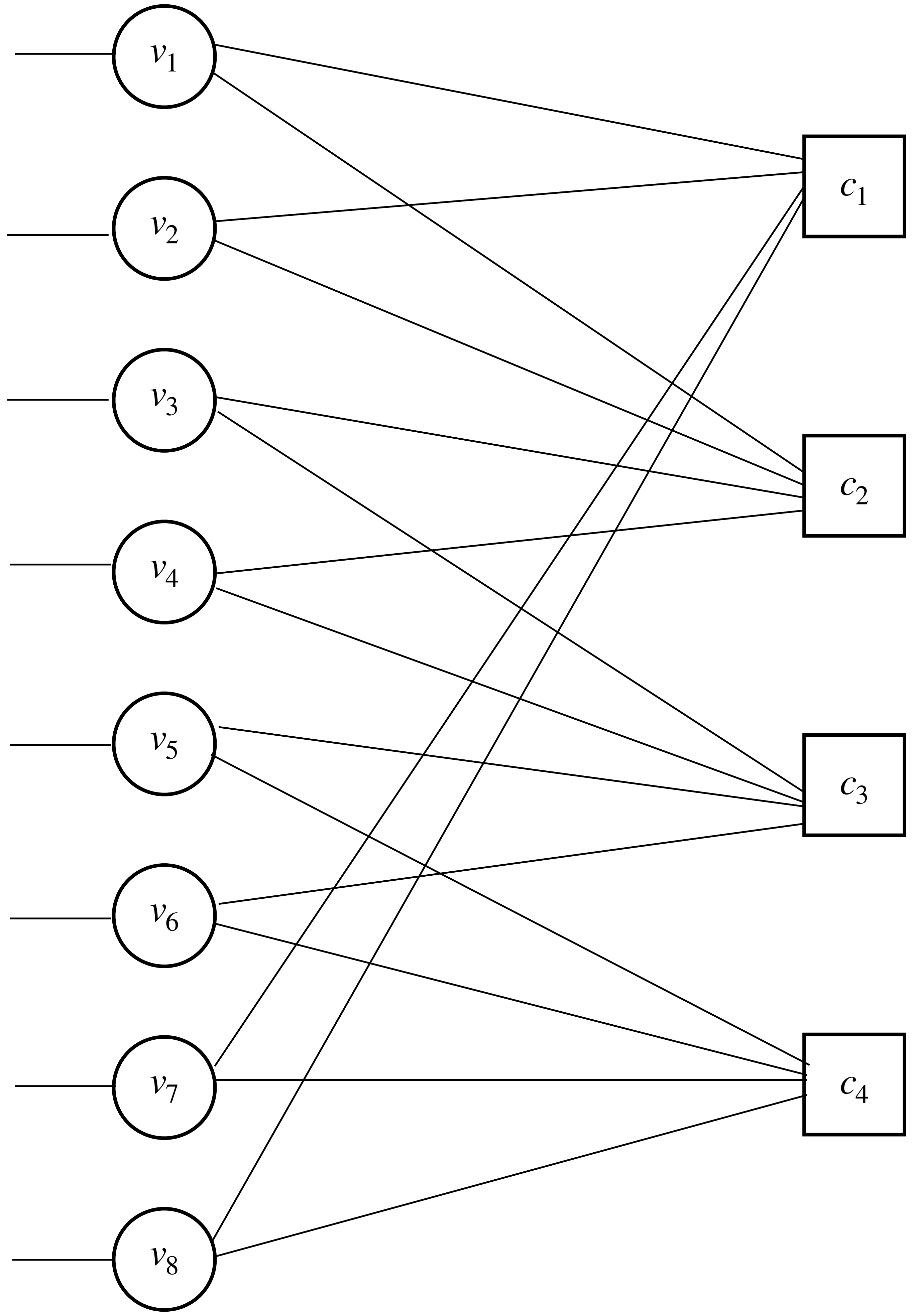
Aggregate information from all *other* variable nodes

- Any other incoming messages are *, send *



LDPC decoding—message passing

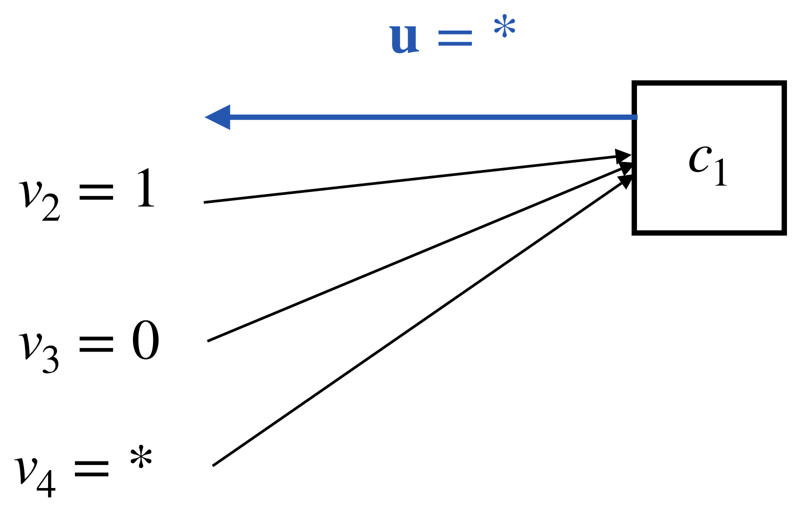
BEC as example



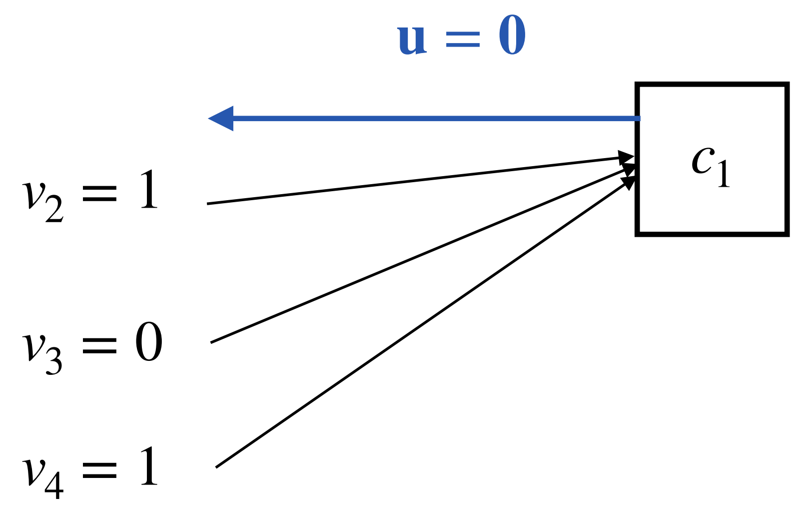
Check to variable message

Aggregate information from all *other* variable nodes

- Any other incoming messages are *, send *



- All other incoming messages are 0 or 1, send XOR of all other incoming messages



BP threshold—density evolution

BEC as example

Finding the largest erasure probability $\varepsilon^\star$ (**threshold**) such that using a BEC with erasure probably $\varepsilon < \varepsilon^\star$, we can reliably transmit the LDPC code with (λ, ρ)-Tanner graph for large enough block length.

BP threshold—density evolution

BEC as example

Finding the largest erasure probability $\varepsilon^\star$ (**threshold**) such that using a BEC with erasure probably $\varepsilon < \varepsilon^\star$, we can reliably transmit the LDPC code with (λ, ρ)-Tanner graph for large enough block length.

Assume cycle-free, so the messages are independent

$p_\ell := \Pr\{v = * \text{ at round } \ell\},\ p_0 = \varepsilon$ the erasure probability

<Remark> Recall that v denotes a variable-to-check message.

Density evolution gives a recursion expression of p_ℓ in terms of $p_{\ell-1}$

The **threshold** $\varepsilon^\star$ is defined as the largest erasure probability such that

$$\lim_{\ell \to \infty} p_\ell = 0$$

For more detailed analysis, we refer the readers to [1], [2].

1. Luby, Michael, Michael Mitzenmacher, and Mohammad Amin Shokrollahi. "Analysis of Random Processes via And-Or Tree Evaluation." *SODA*. Vol. 98. 1998.
2. Richardson, Thomas J., and Rüdiger L. Urbanke. "The capacity of low-density parity-check codes under message-passing decoding." *IEEE Transactions on information theory* 47.2 (2001): 599-618.

Asymptotic analysis—density evolution

BEC as example

For a variable node with degree d

$\Pr\{v = * \text{ at round } \ell \mid \text{node degree} = d\}$

$= \Pr\{\text{all other incoming messages are } * \text{ at round } \ell - 1\}$

$= \Pr\{u_0 = *\}\Pr\{\forall i \in [d-1],\ u_i = * \text{ at round } \ell - 1\}$

$= p_0 \Pr\{u = * \text{ at round } \ell - 1\}^{d-1}$

$$p_\ell := \Pr\{v = * \text{ at round } \ell\} = \sum_{i=1}^{d_v} \lambda_i p_0 \Pr\{u = * \text{ at round } \ell - 1\}^{d-1}$$

$$= p_0 \lambda(\Pr\{u = * \text{ at round } \ell - 1\})$$

Next we need to analyze $\Pr\{u = * \text{ at round } \ell - 1\}$

Asymptotic analysis—density evolution

BEC as example

For a check node with degree d

$\Pr\{u = * \text{ at round } \ell \mid \text{node degree} = d\}$

$= \Pr\{\text{exists some other incoming message is } * \text{ at round } \ell - 1\}$

$= 1 - \Pr\{\forall i \in [d-1],\ v_i \neq * \text{ at round } \ell - 1\}$

$= 1 - (1 - \Pr\{v = * \text{ at round } \ell - 1\})^{d-1}$

$$
\begin{aligned}
\Pr\{u = * \text{ at round } \ell\} &= \sum_{i=2}^{d_c} \rho_i (1 - \Pr\{u = * \text{ at round } \ell - 1\})^{d-1} \\
&= 1 - \rho(1 - \Pr\{u = * \text{ at round } \ell - 1\}) \\
&= 1 - \rho(1 - p_{\ell-1})
\end{aligned}
$$

BP threshold—density evolution

BEC as example

By analyzing $\Pr\{u = * \text{ at round } \ell - 1\}$ and $\Pr\{v = * \text{ at round } \ell - 1\}$, We obtain the recursion $\mathbf{p}_\ell = \varepsilon\lambda(\mathbf{1} - \rho(\mathbf{1} - \mathbf{p}_{\ell-1}))$

To ensure $\lim_{\ell\to\infty} p_\ell = 0$, we need $p_\ell < (1-\delta)p_{\ell-1}$ for all ℓ

This leads to finding the largest $\varepsilon^\star$ such that

$$\varepsilon^\star\lambda(1 - \rho(1 - x)) - x < 0.$$

There are close-form solutions for regular LDPC codes.

BP threshold—threshold saturation

In general, BP decoding is *weaker* than MAP decoder. (Example provided in the next section.)

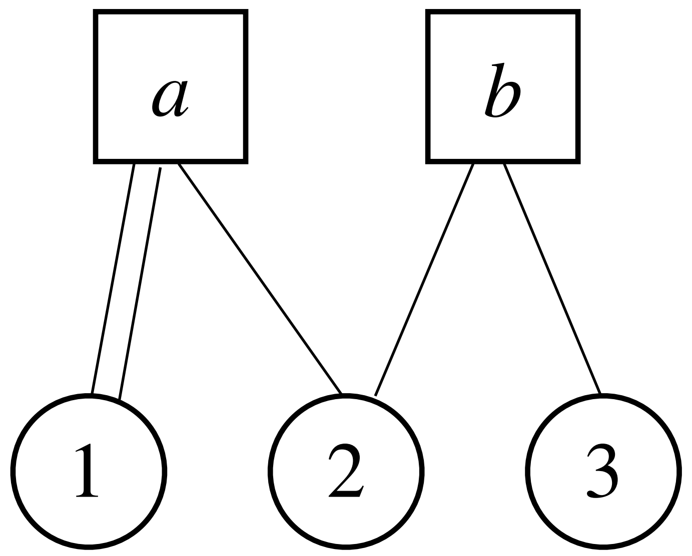
Threshold saturation: When BP threshold and MAP threshold coincide.

SC-LDPC exhibits the threshold saturation phenomena on BEC[3] and BMS[4] channels!

3. Kudekar, Shrinivas, Thomas J. Richardson, and Rüdiger L. Urbanke. "Threshold saturation via spatial coupling: Why convolutional LDPC ensembles perform so well over the BEC." *IEEE Transactions on Information Theory* 57.2 (2011): 803-834.
4. Kudekar, Shrinivas, Tom Richardson, and Rüdiger L. Urbanke. "Spatially coupled ensembles universally achieve capacity under belief propagation." *IEEE Transactions on Information Theory* 59.12 (2013): 7761-7813.

LDPC construction—protograph

M-lifting: "Copy and permute."



Protograph with protomatrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

LDPC construction—protograph

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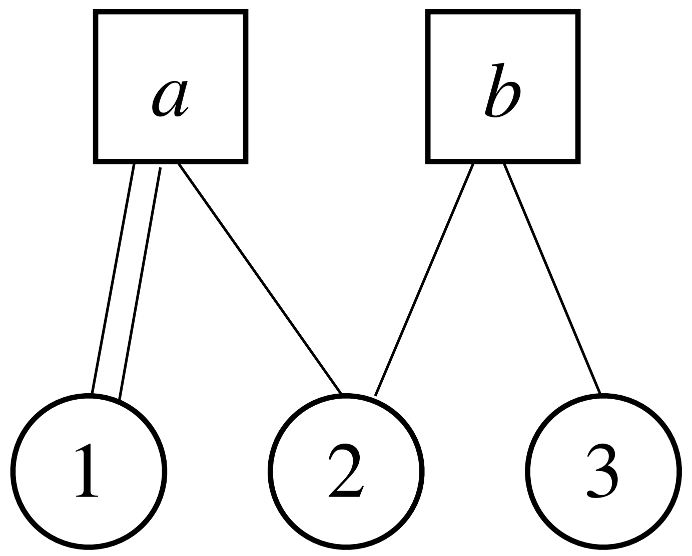

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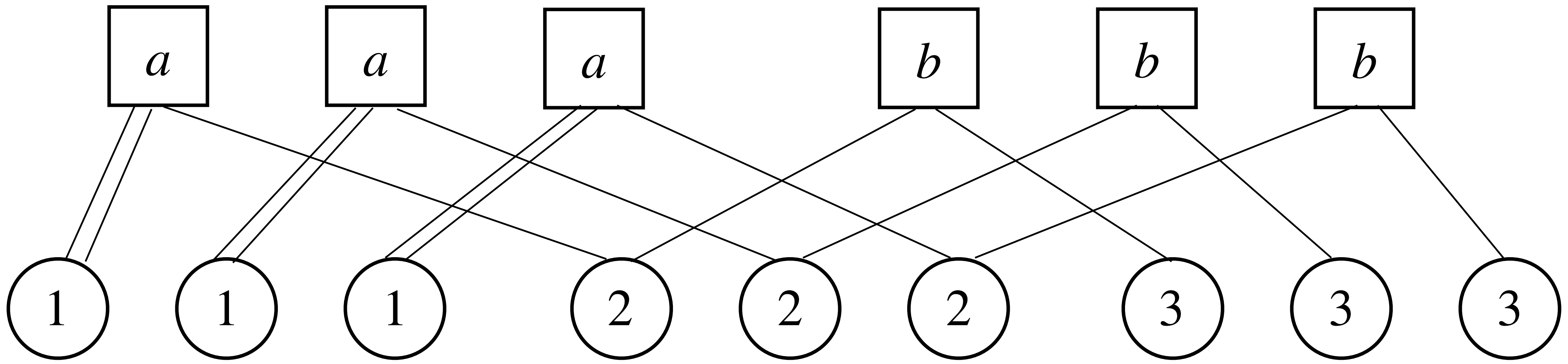

Copy *M* times
(in this example $M = 3$)

LDPC construction—protograph

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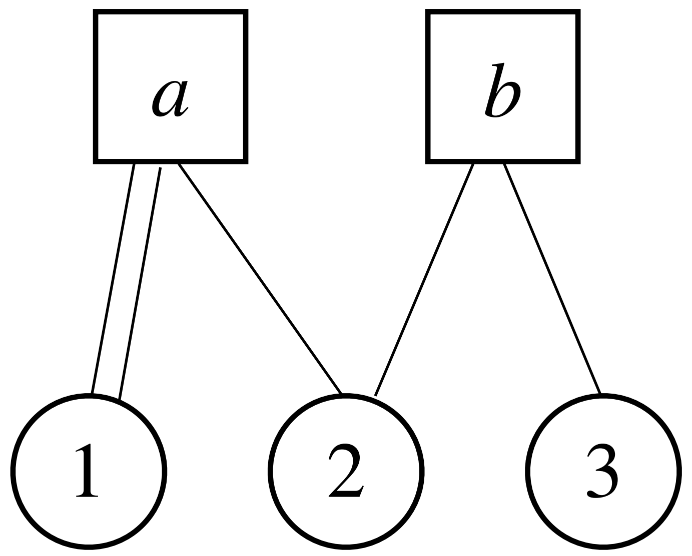

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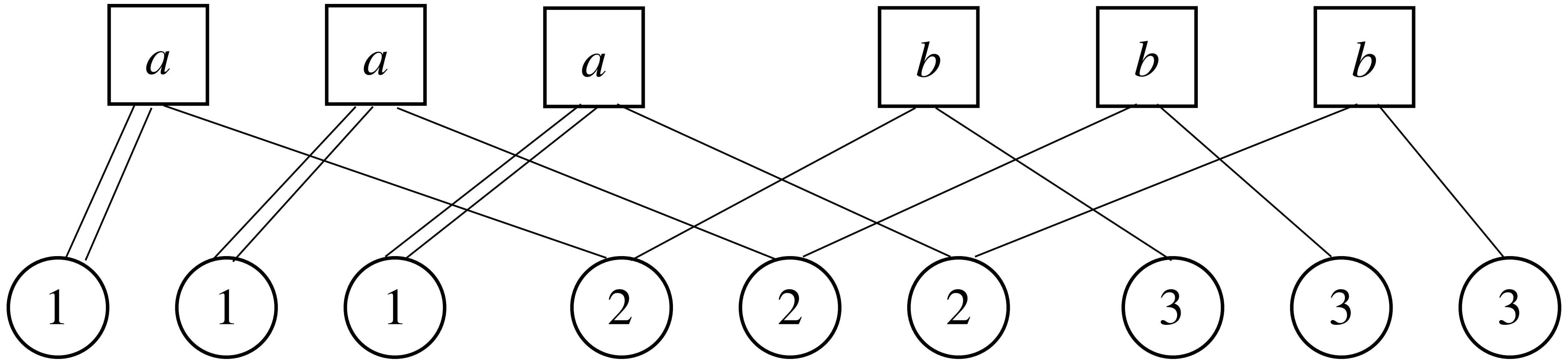

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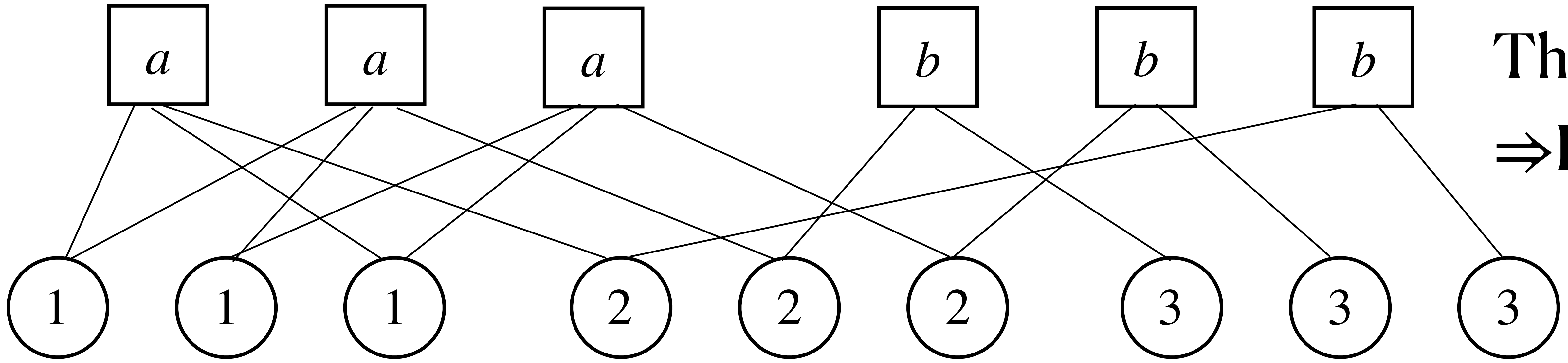

The above graph is disconnected

⇒**Permute** the edges

LDPC construction—protograph

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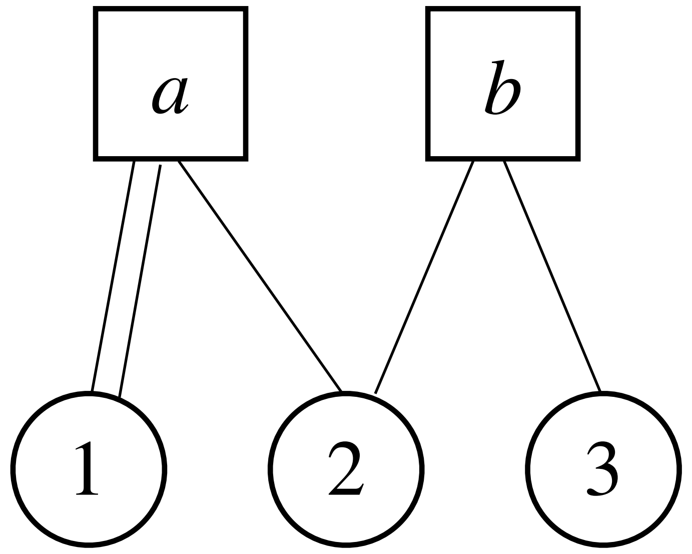
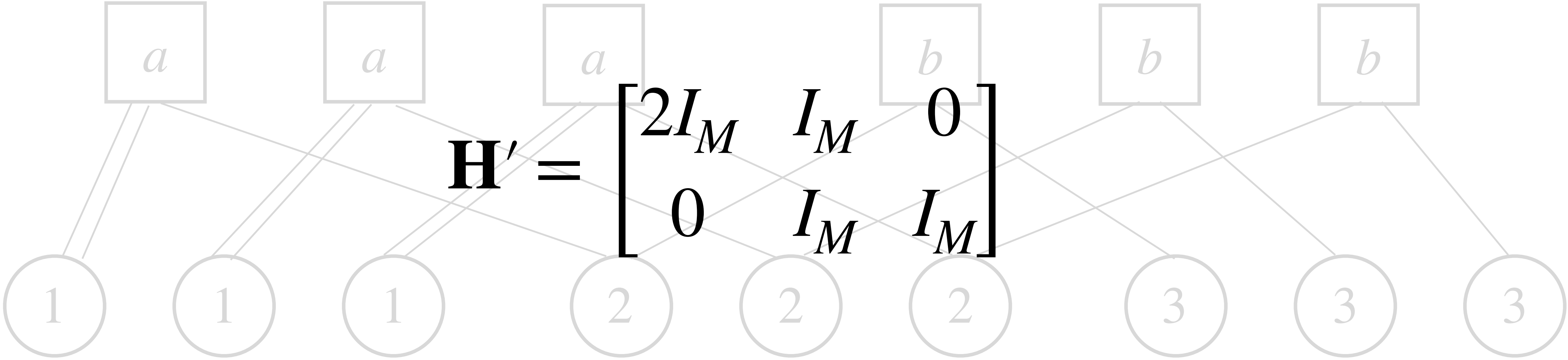
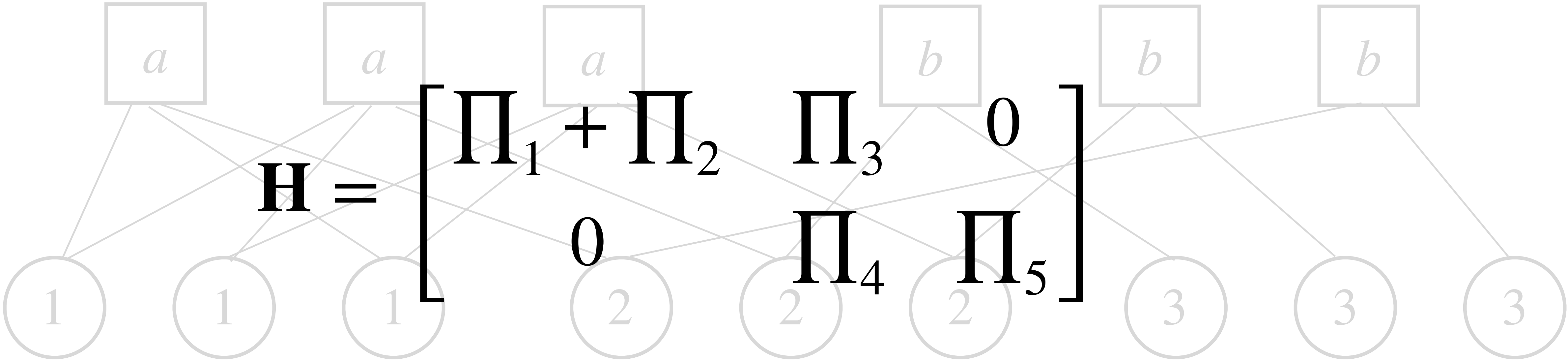


Protograph with protomatrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$



$$\mathbf{H}' = \begin{bmatrix} 2I_M & I_M & 0 \\ 0 & I_M & I_M \end{bmatrix}$$

Copy M times
(in this example $M = 3$)

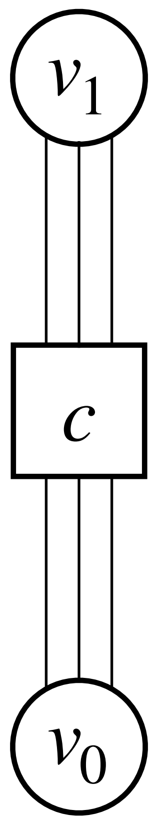


$$\mathbf{H} = \begin{bmatrix} \prod_1 + \prod_2 & \prod_3 & 0 \\ 0 & \prod_4 & \prod_5 \end{bmatrix}$$

Permute the edges
(Here $\prod_i$'s denote arbitrary permutation matrices)

Spatially Coupled-LDPC

Edge spreading: "Copy and spread."

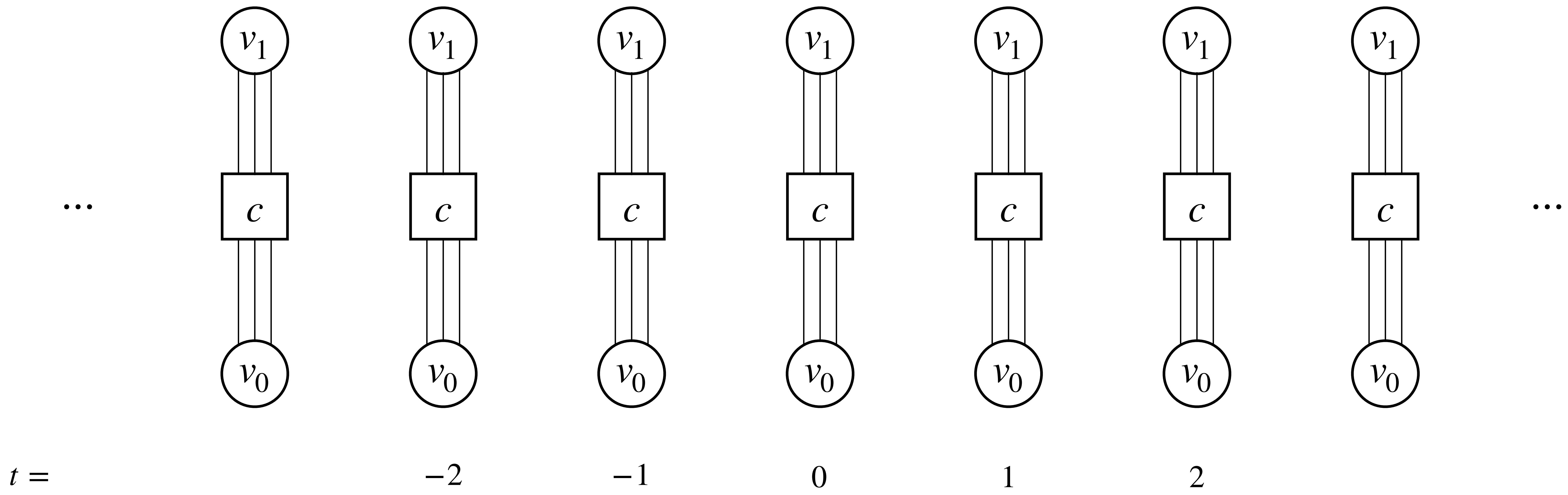


(3,6)−protograph

Spatially Coupled-LDPC

Edge spreading: “Copy and spread.”

Infinite disjoint copies

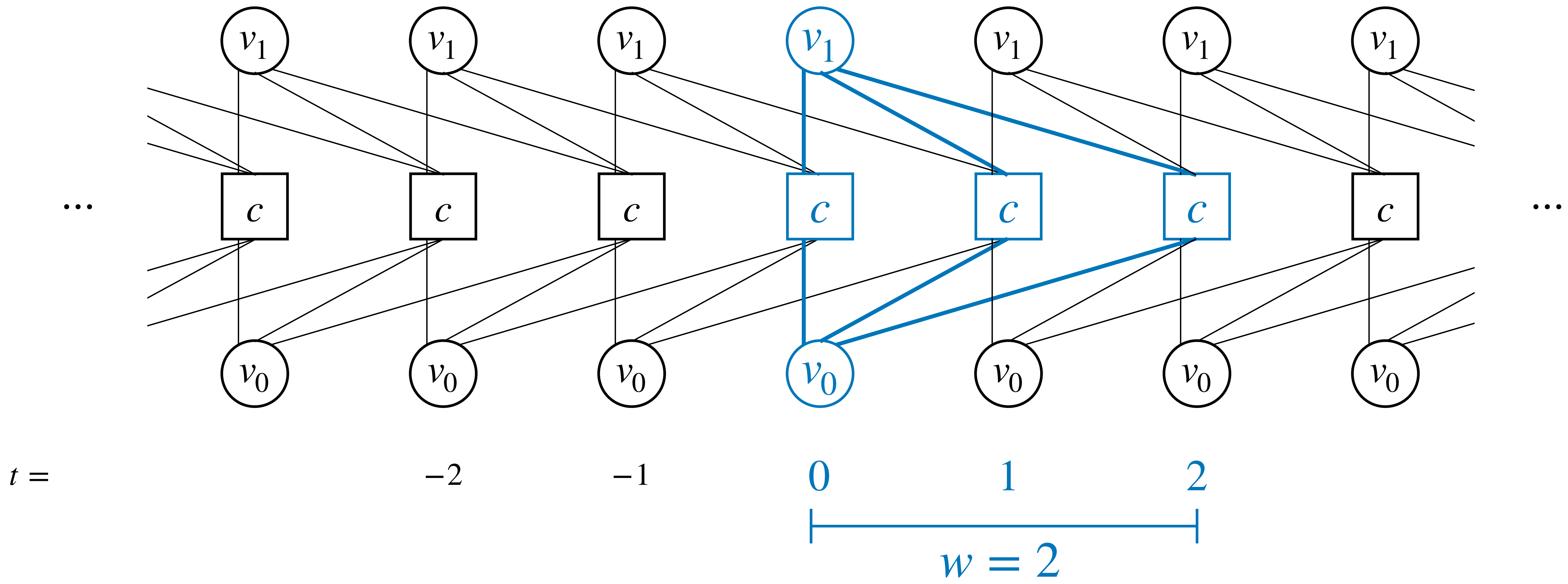


(3,6)−protograph

Spatially Coupled-LDPC

Edge spreading: "Copy and spread."

Infinite disjoint copies



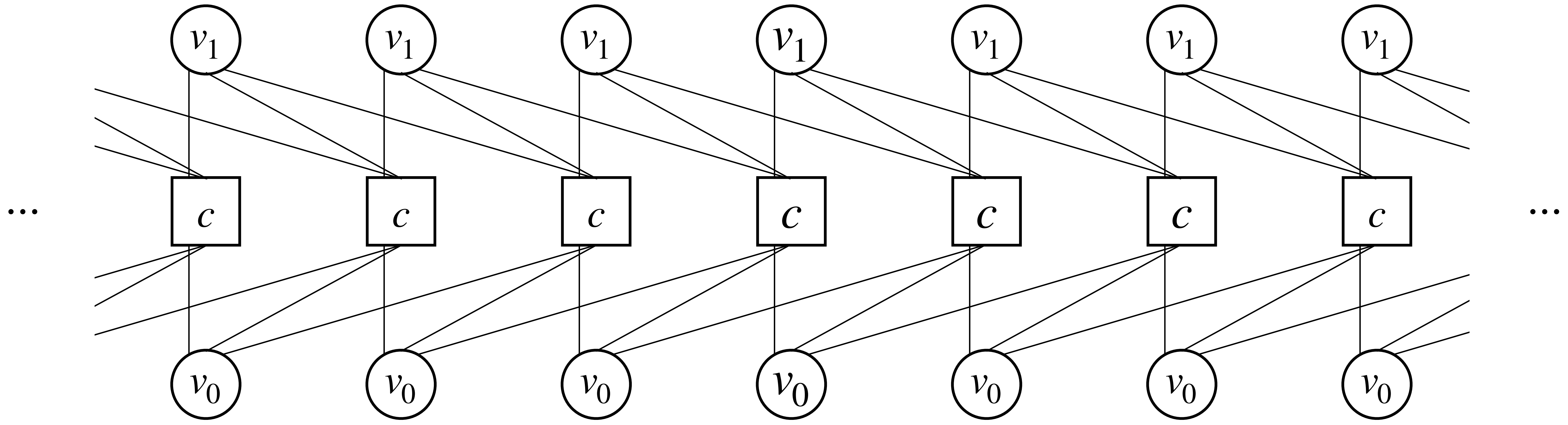
Spread the edges. Only forward in t.

Coupling width (w): the farthest check node that an edge can spread out to.

(In this example $w = 2$.)

Spatially Coupled-LDPC

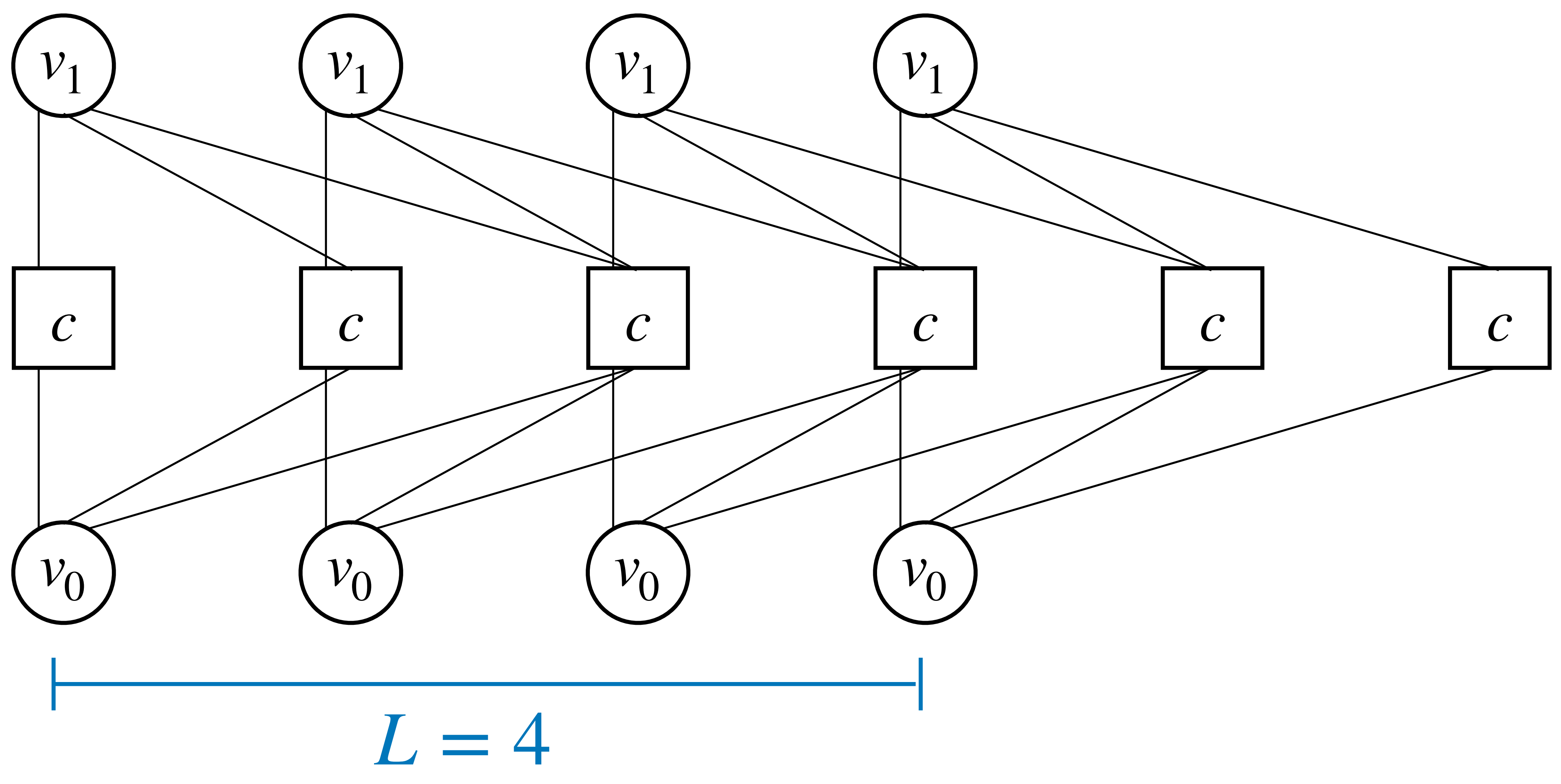
Truncation



Truncate.
Keep L copies of variable nodes, all of their edges, and all of the check nodes they are connected to.
Coupling length (L): the number of copies of variable nodes

Spatially Coupled-LDPC

Truncation

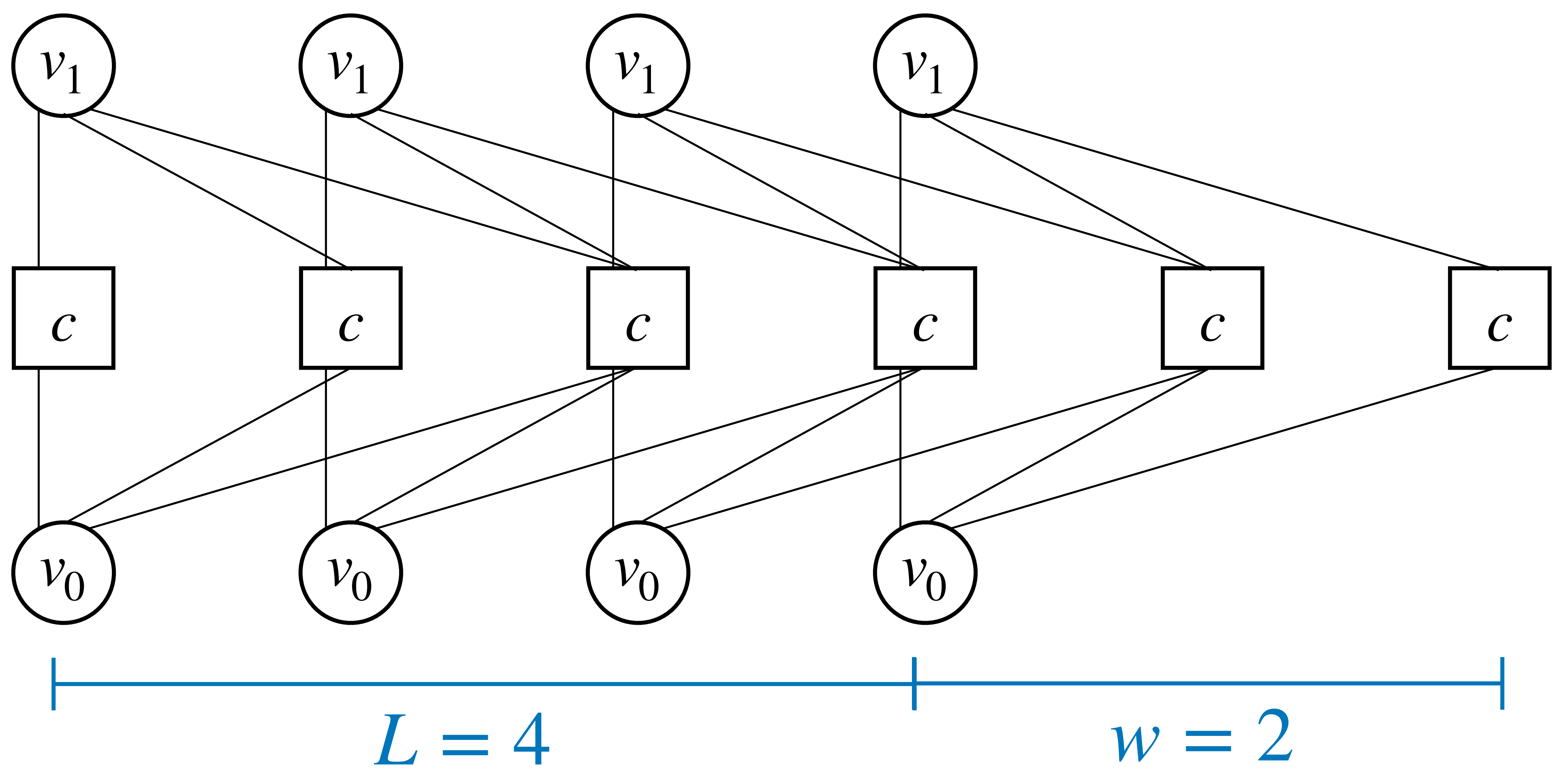


Truncate.
Keep L copies of variable nodes, all of their edges, and all of the check nodes they are connected to.
Coupling length (L): the number of copies of variable nodes
(In this example $L = 4$.)

Spatially Coupled-LDPC

A special class of SC-LDPC: $\mathscr{C}(j, k, L)$ SC-LDPC-BC

(3,6)−protograph



(j, k): the parameter of the base protograph.
(In this example $(j, k) = (3,6)$)

w: the coupling width is chosen as $\gcd(j, k) - 1$
(In this example $w = \gcd(3,6) - 1 = 2$)

The above figure is a $\mathscr{C}(3,6,4)$ SC-LDPC-BC.

SC-LDPC vs LDPC

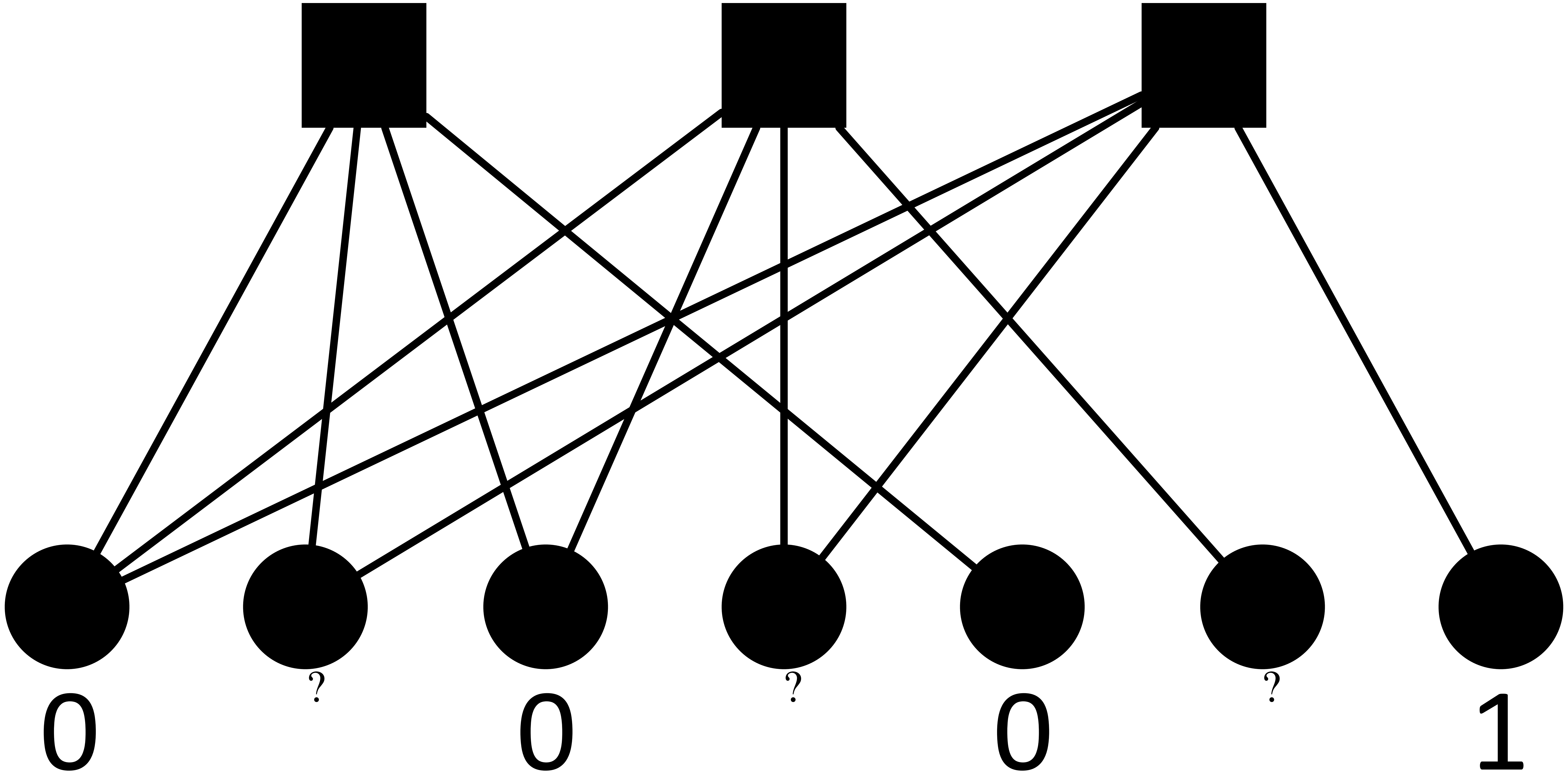
- Pros
  - Better BP thresholds
  - Low error floor
  - Good at burst error correction
- Cons
  - Higher decoding latency
  - Increase decoding complexity
  - Both can be mitigated by slide window decoding
- Applications in 5G, distributed storage, burst error channel…

Abdoul-Hadi Konfé, Pasteur Poda, Raphaël Le Bidan. Design Techniques of Spatially Coupled Low-Density Parity-Check Codes: A Review and Tutorial on 5G New Radio. CARI 2022, Oct 2022, Yaounde, Cameroon.

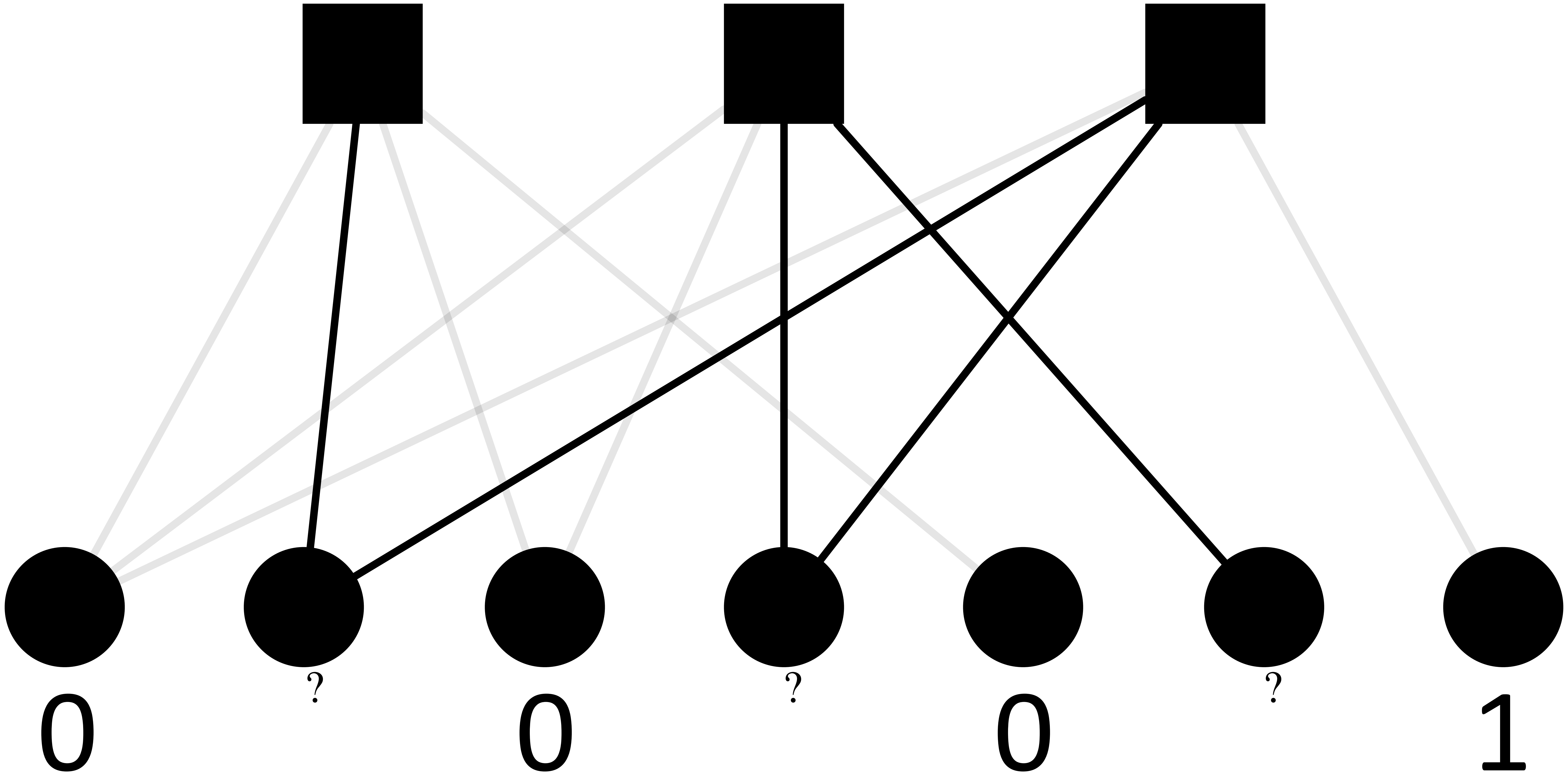
Mitchell, David GM, et al. "Spatially coupled generalized LDPC codes: Asymptotic analysis and finite length scaling." *IEEE Transactions on Information Theory* 67.6 (2021): 3708-3723.

BP thresholds for LDPC & SC-LDPC

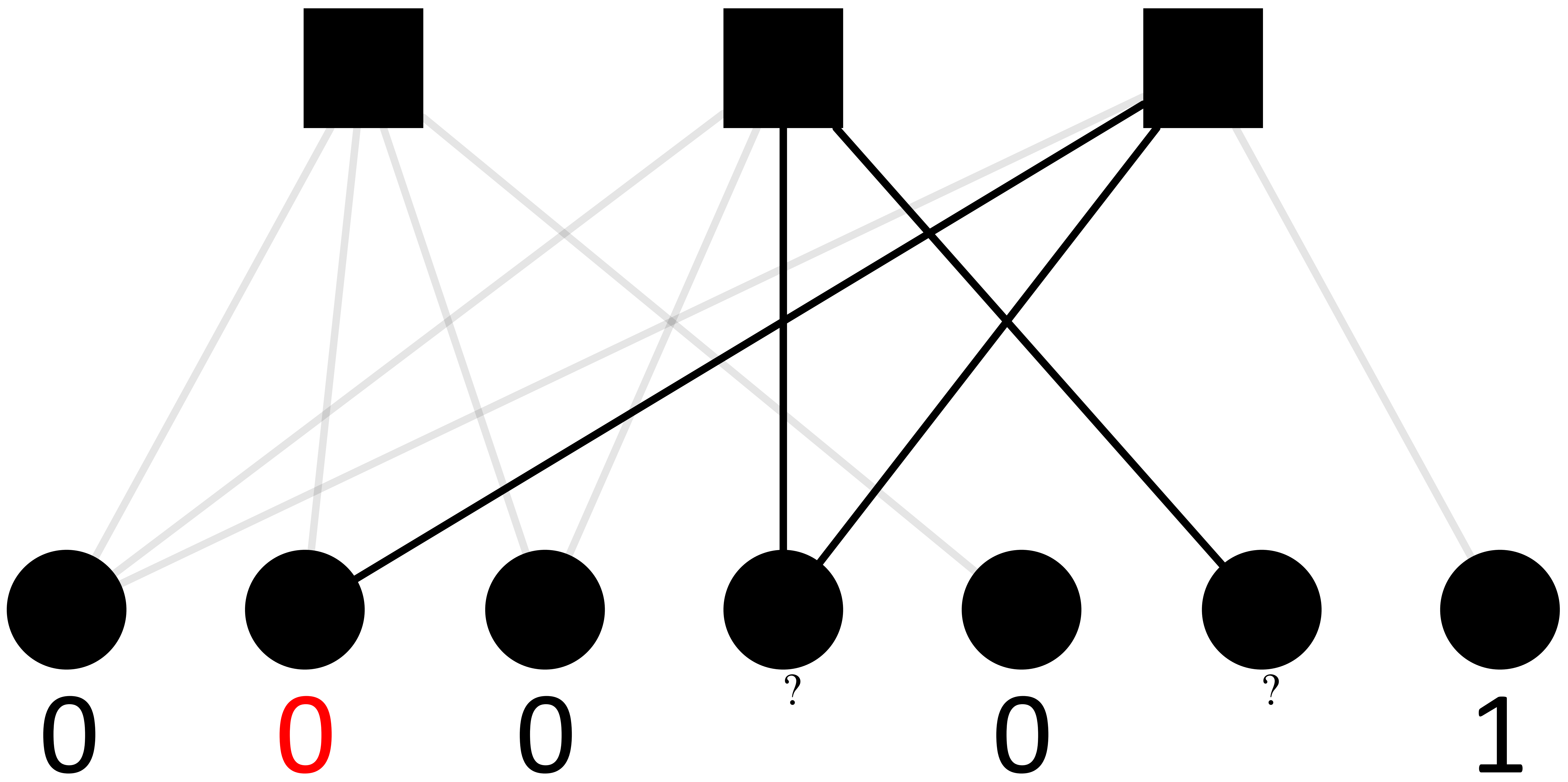
BP for LDPC



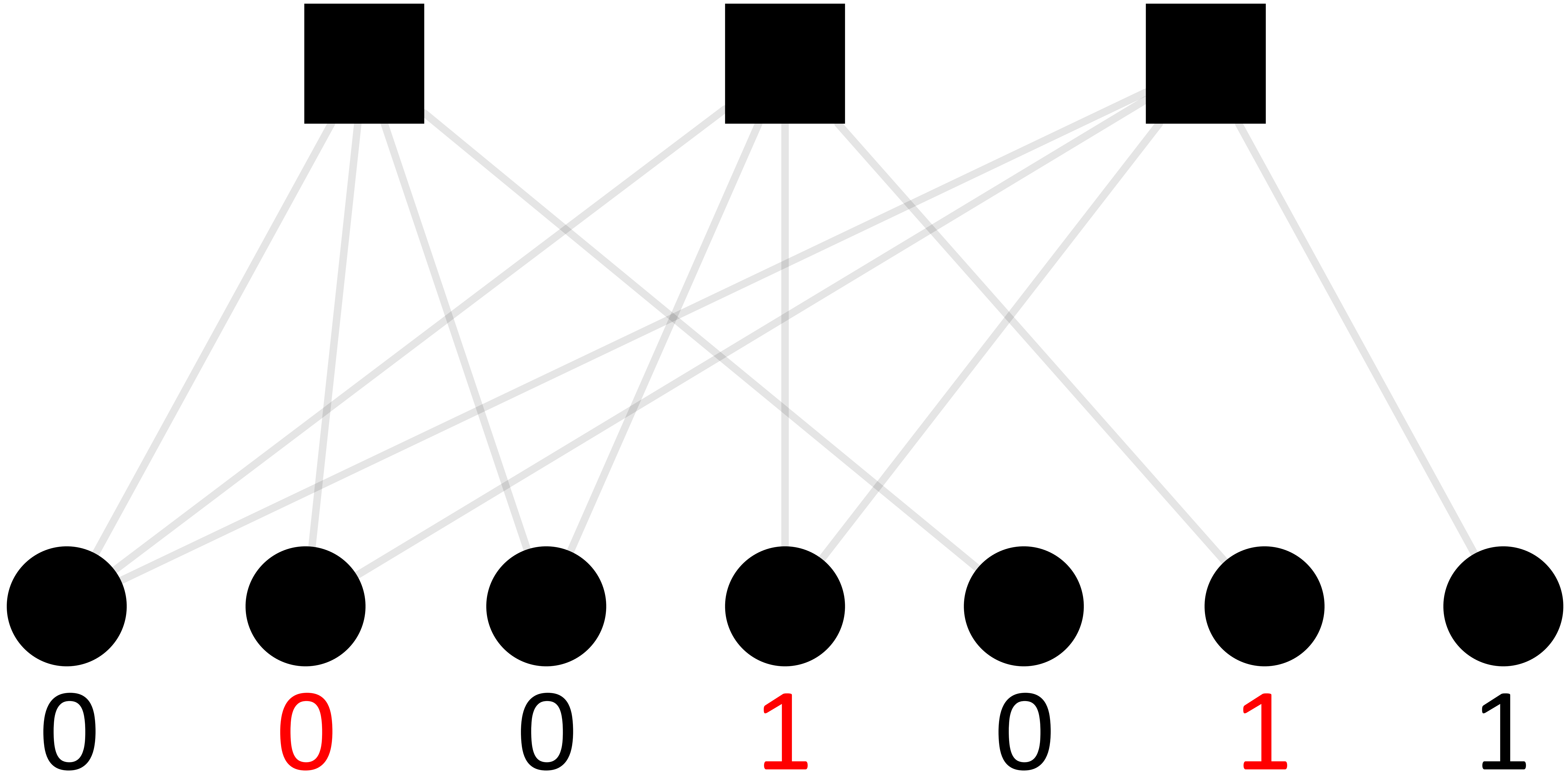
BP for LDPC



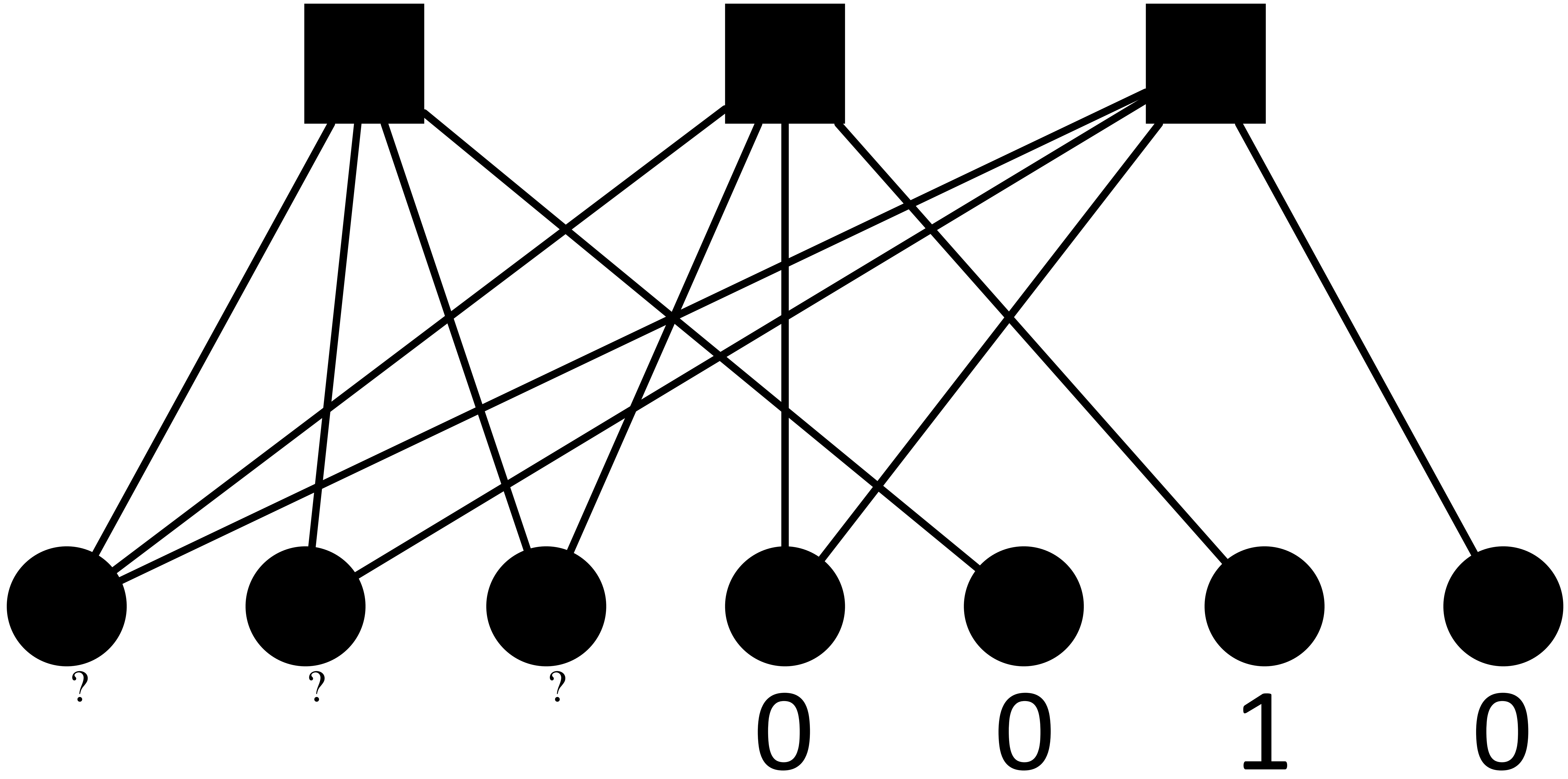
BP for LDPC



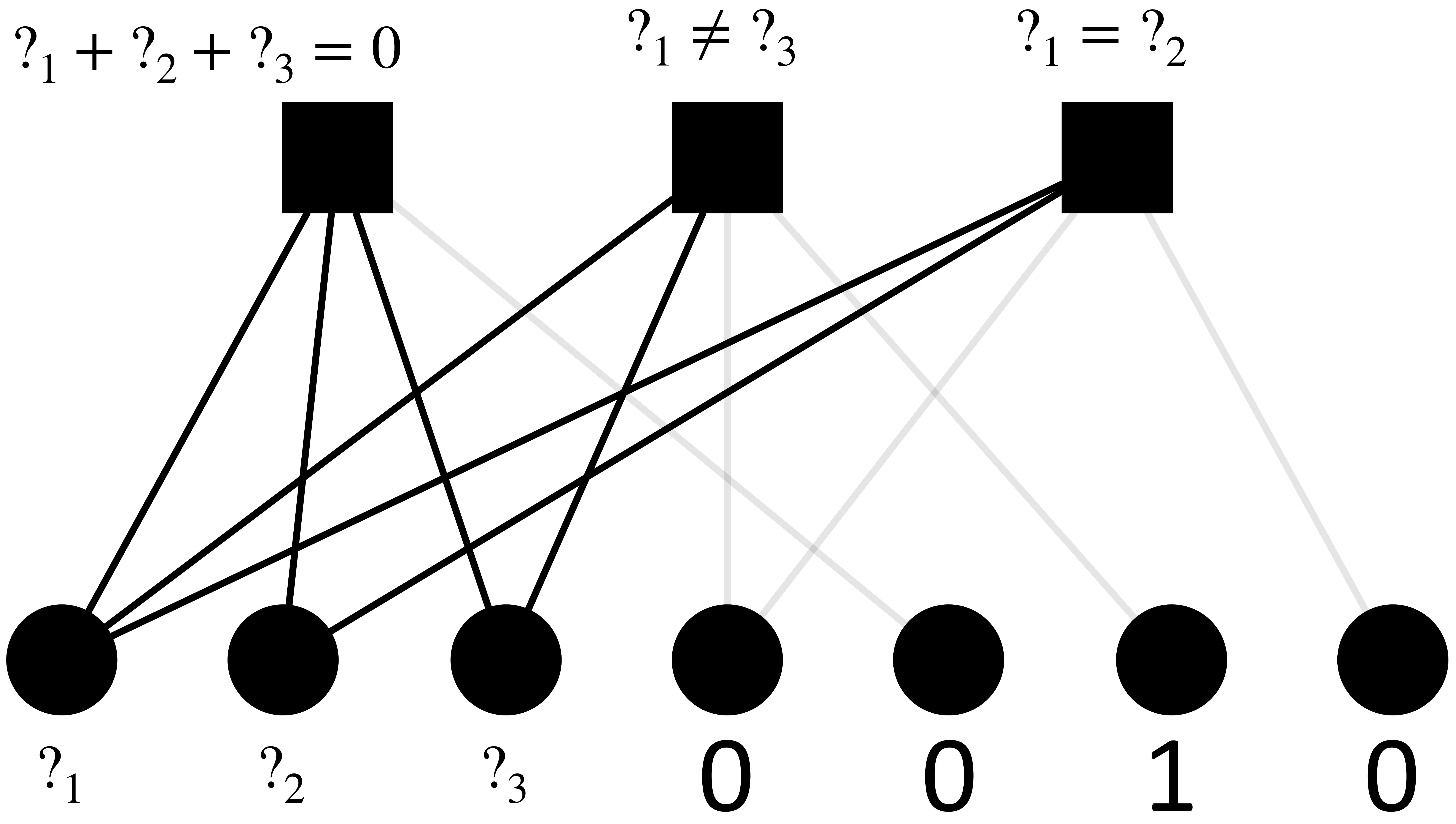
BP for LDPC



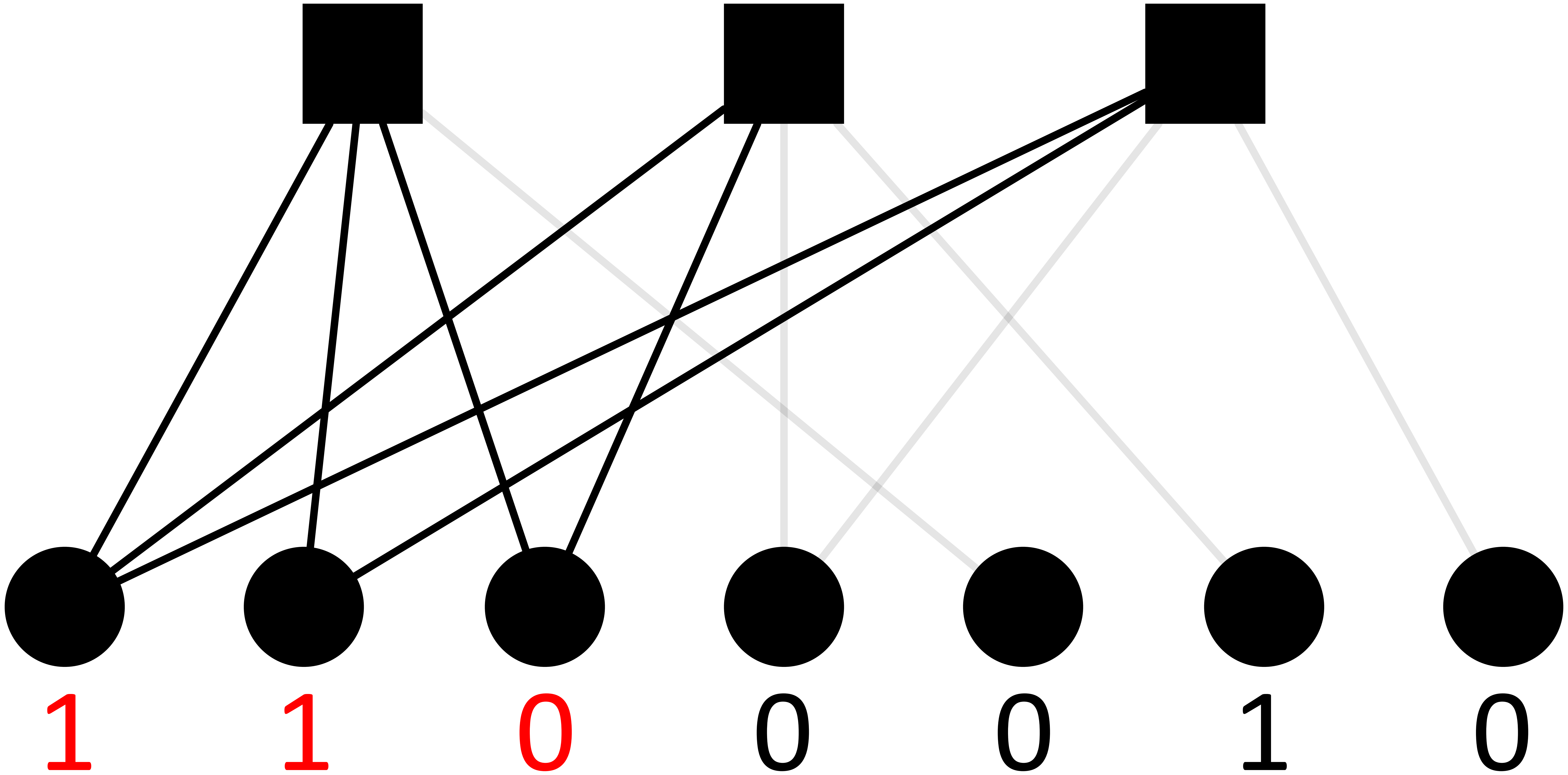
BP is suboptimal



BP is suboptimal



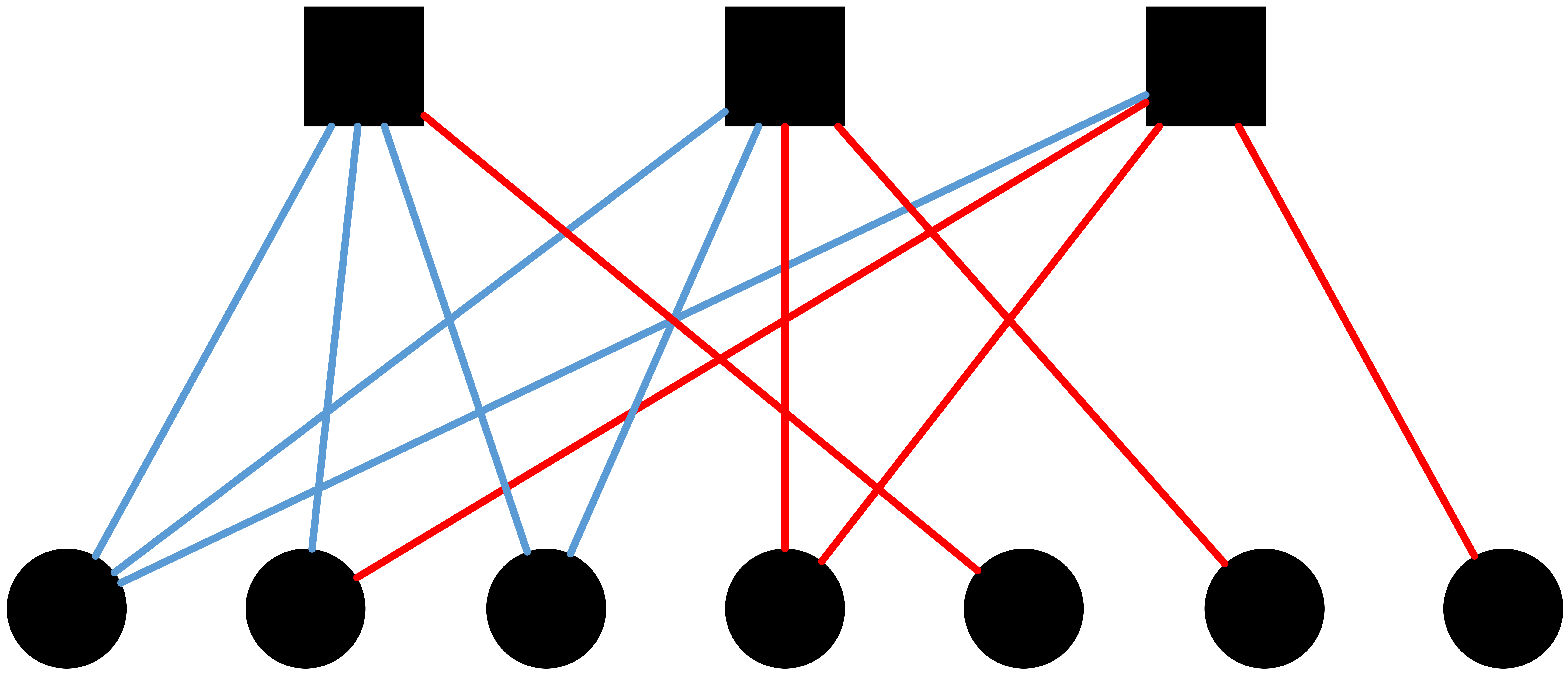
BP is suboptimal



Construct LDPC code
such that
BP decoding algorithm is optimal

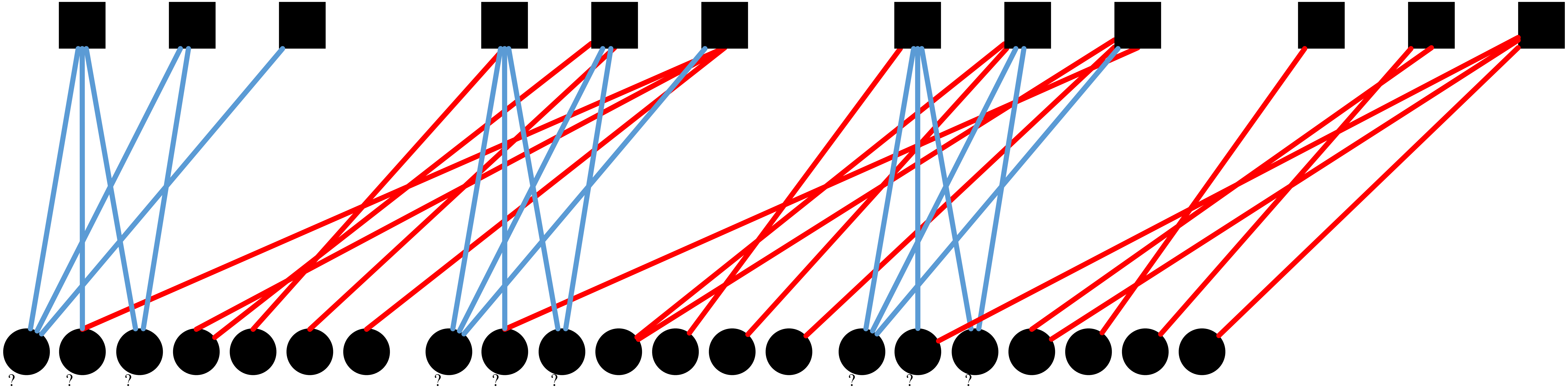
BP for SC-LDPC

- Base graph

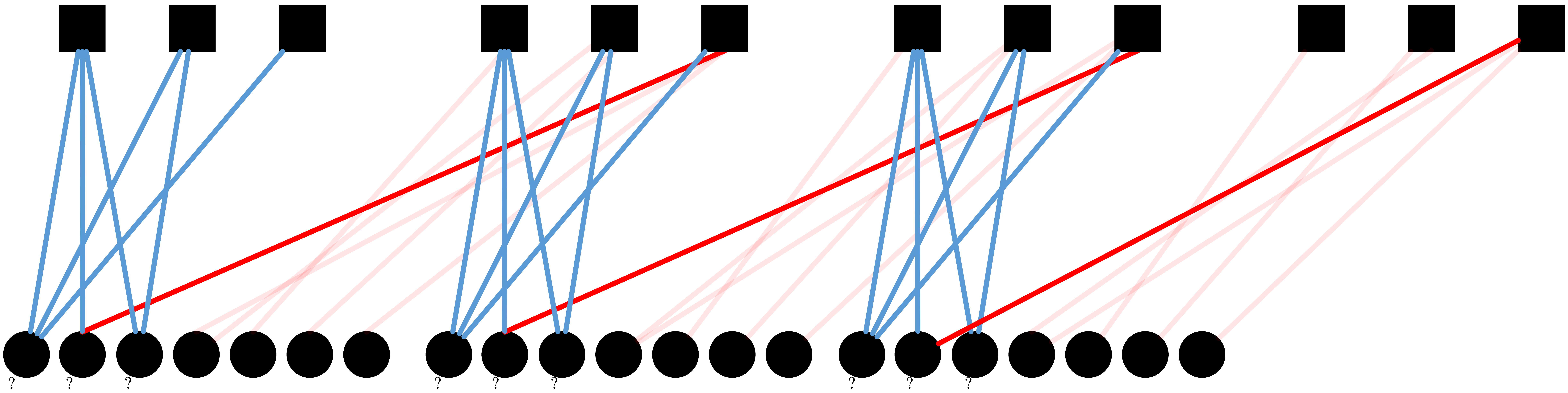


BP for SC-LDPC

- L = 3, W = 1



BP for SC-LDPC



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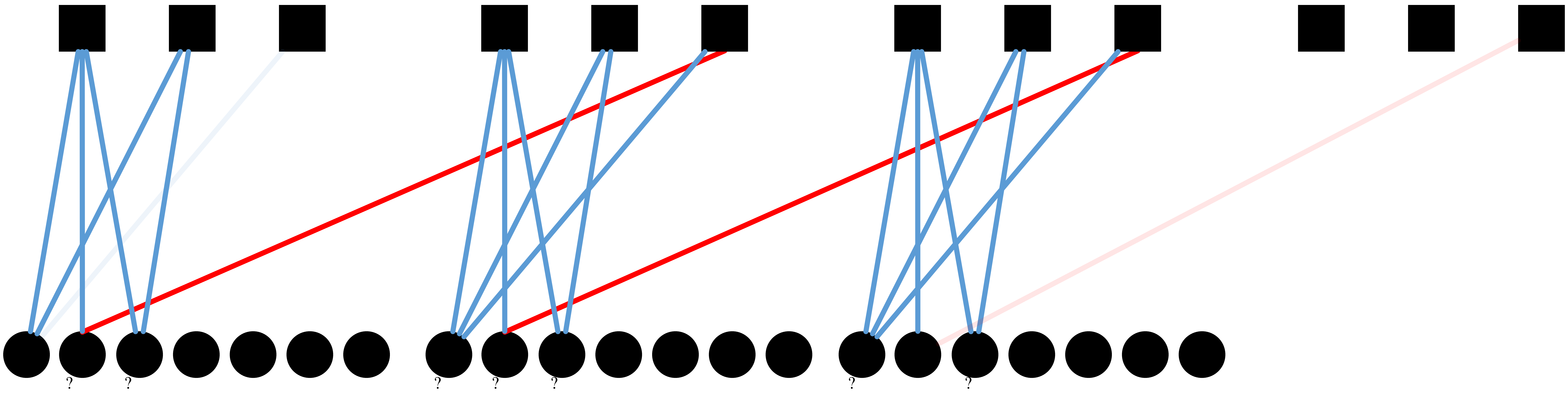

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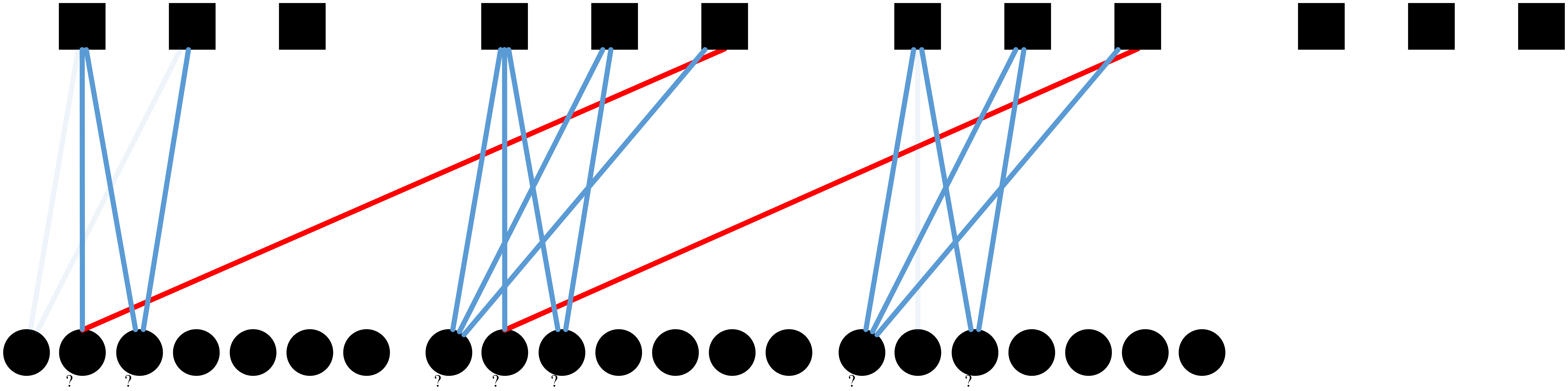

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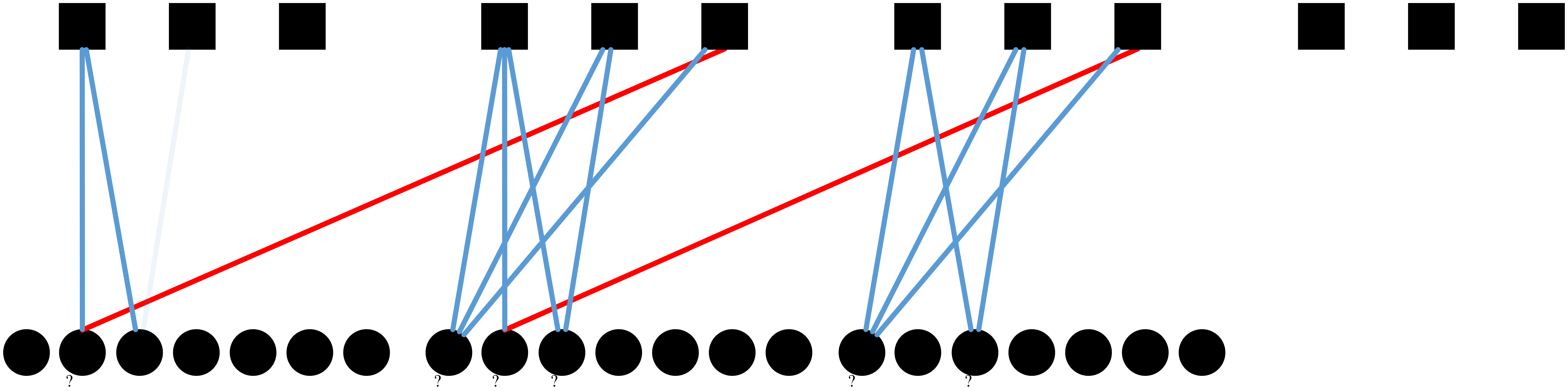

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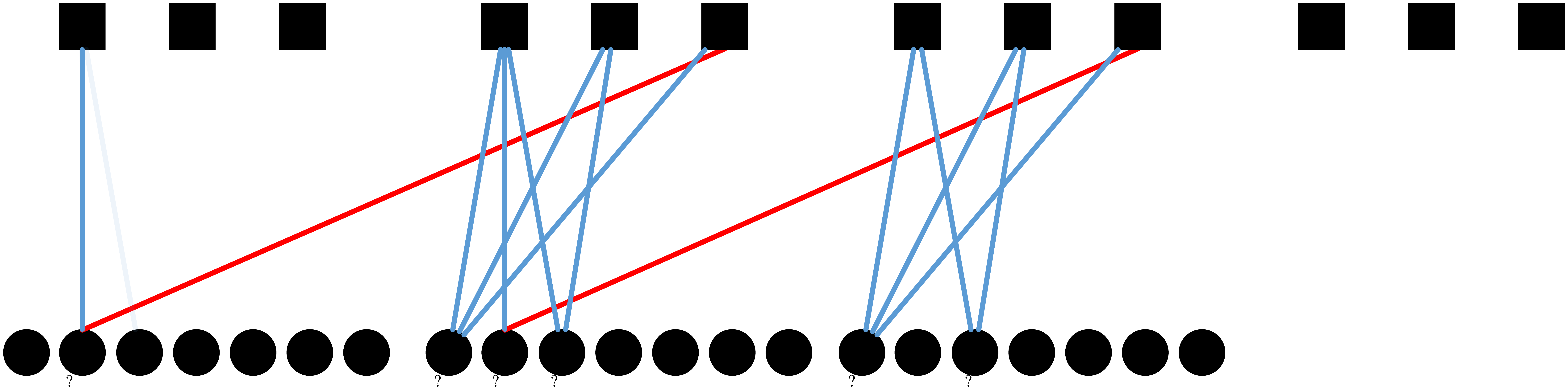

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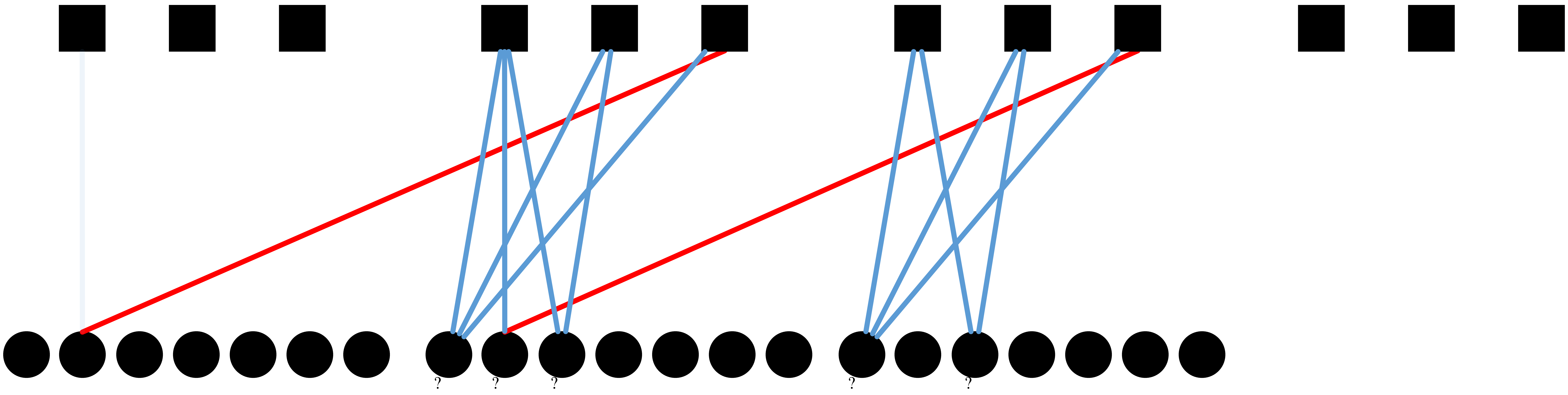
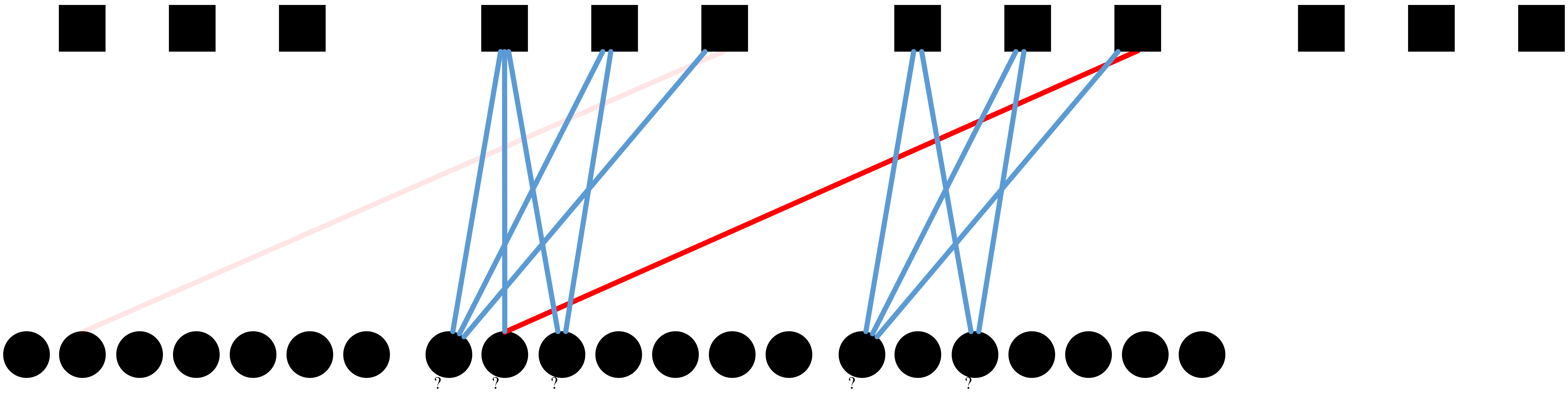
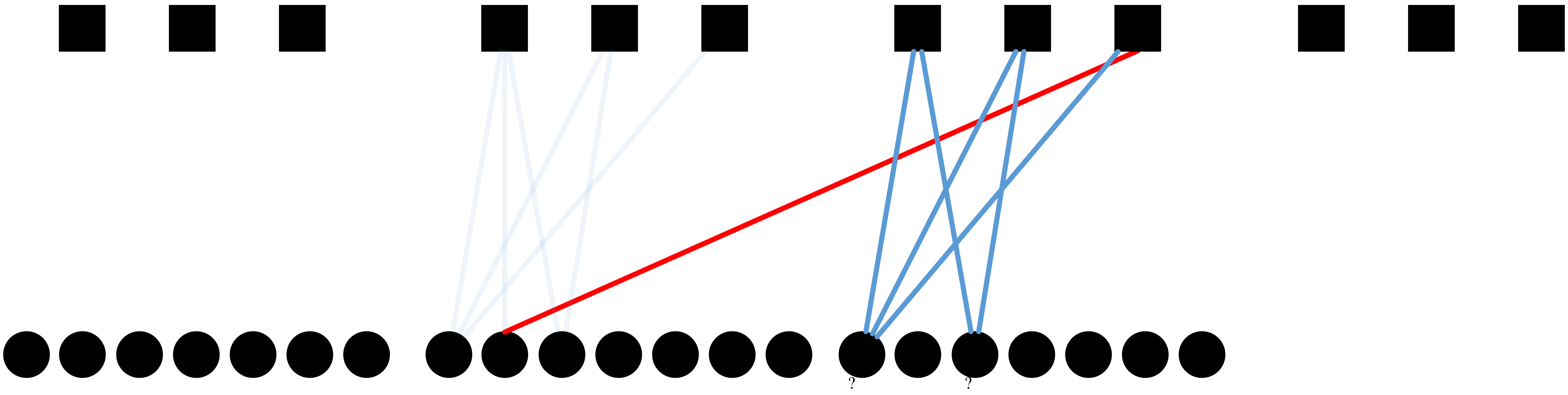


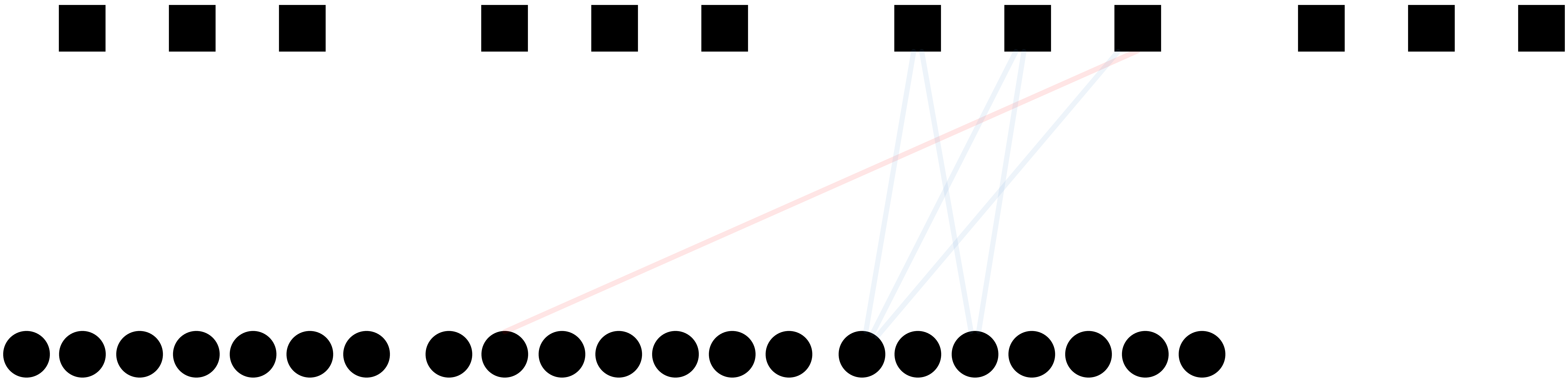
BP for SC-LDPC



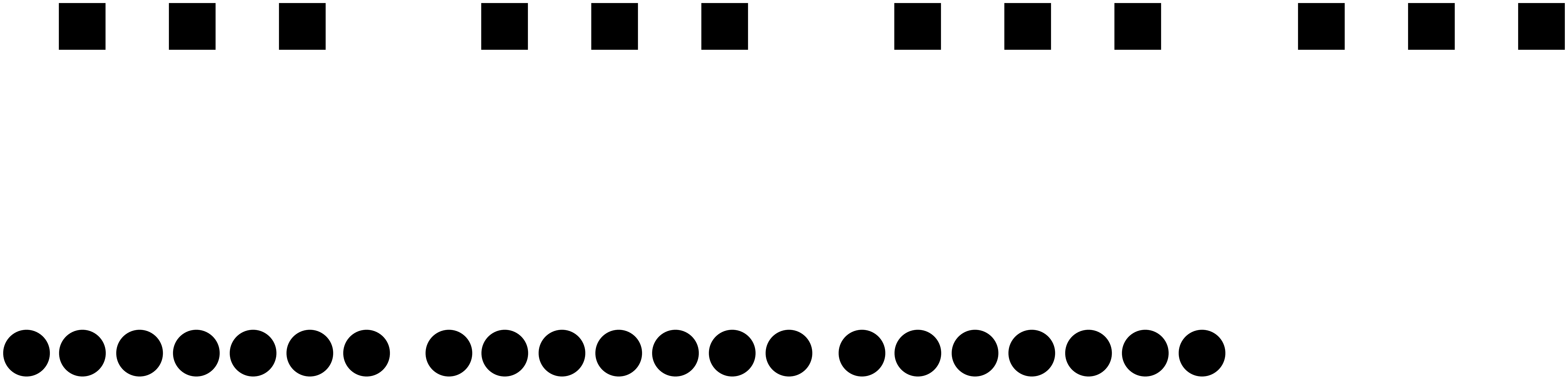
BP for SC-LDPC



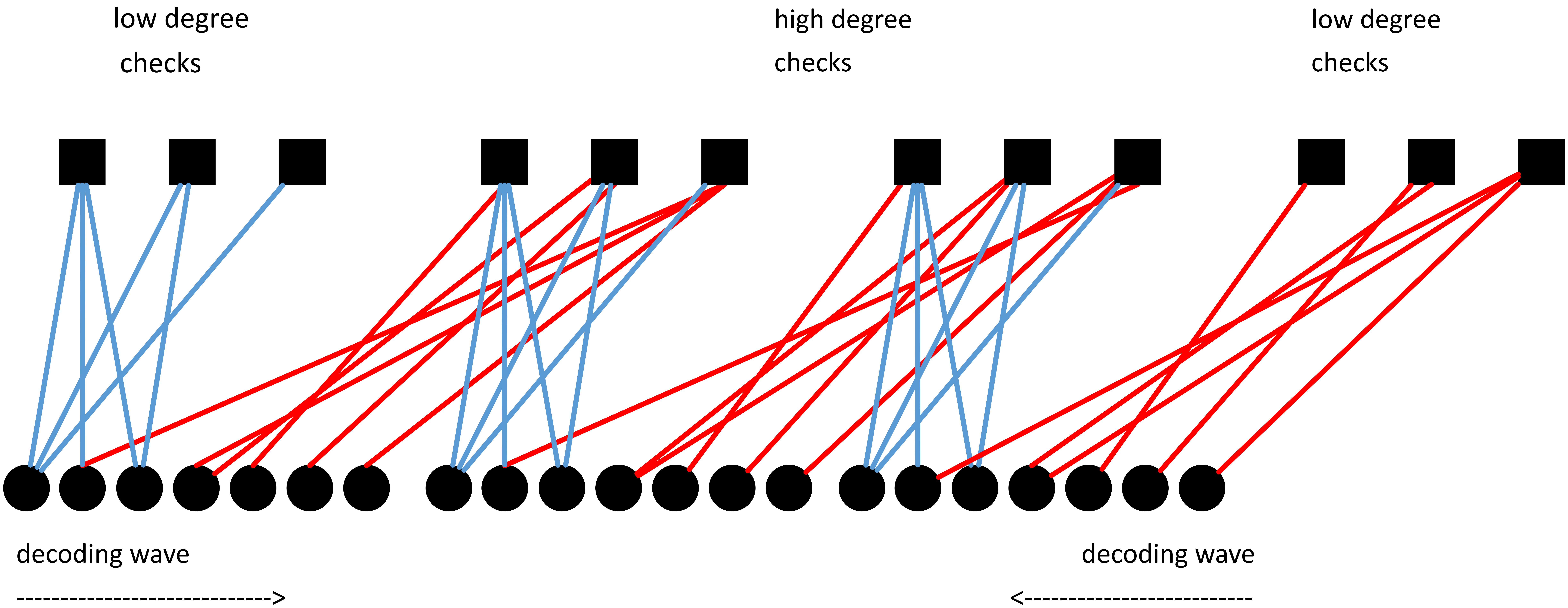
BP for SC-LDPC



BP for SC-LDPC



Decoding ‘wave’



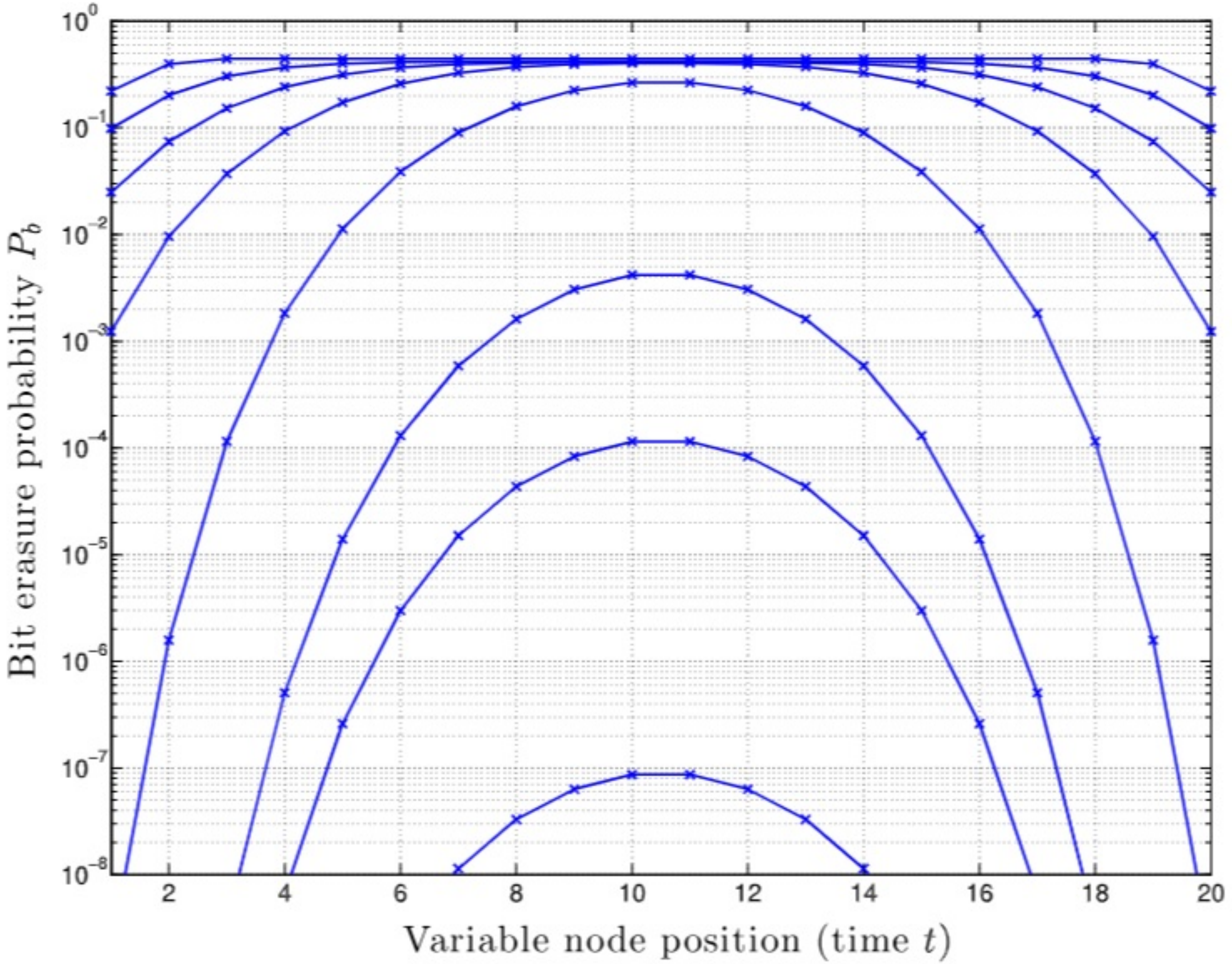
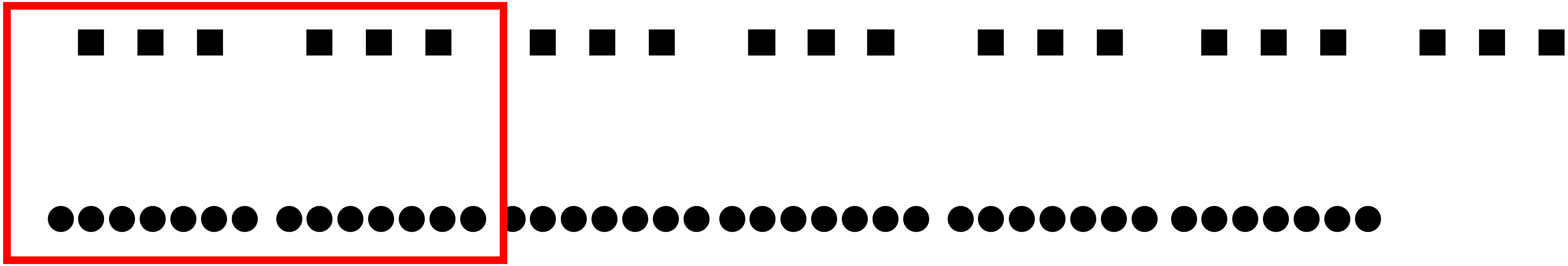
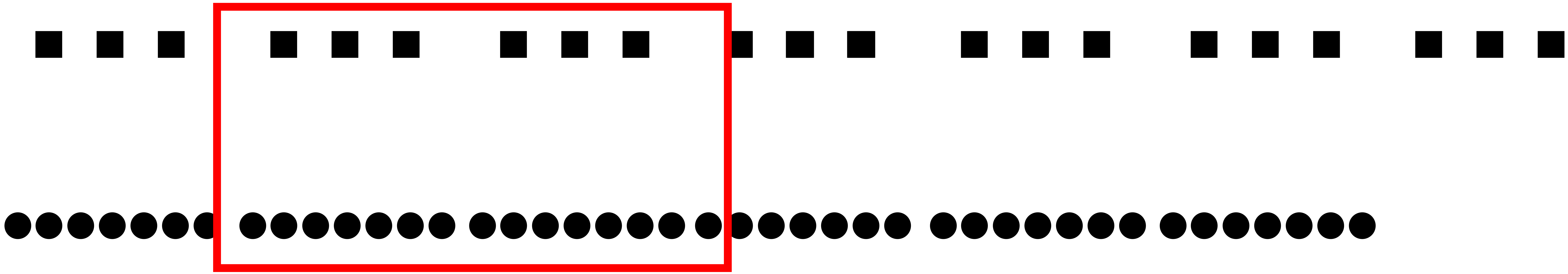


Fig. 7. Evolution of the average bit erasure probability P_b of the variable nodes at time t for the $\mathcal{C}(3,6,20)$ SC-LDPC-BC ensemble transmitted over a BEC with erasure probability $\varepsilon = 0.48$ for iterations $i = 1, 5, 20, 50, 90, 98, 99, 100$ (from top to bottom).

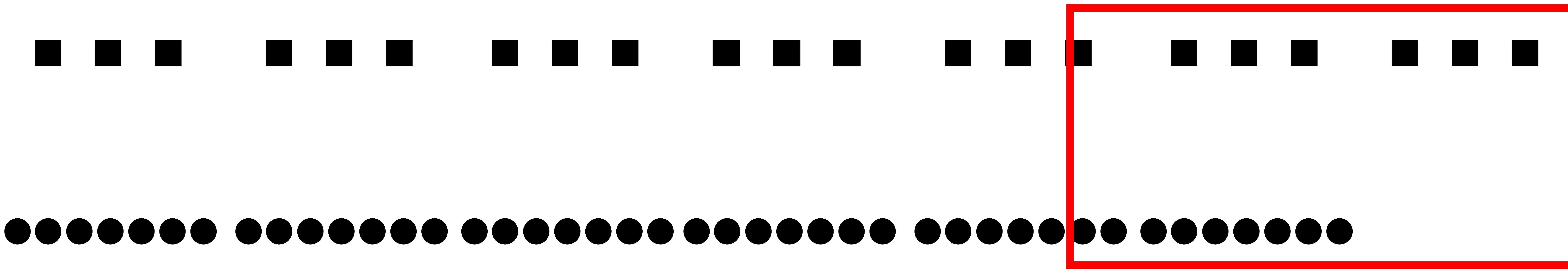
Sliding window decoding



Sliding window decoding



Sliding window decoding



Pros & cons of slide window decoding

- Pros
  - Reduce decoding complexity
  - Low latency
- Cons
  - Increasing error floors
  - Increase # of iterations

Herrmann, Matthias, and Norbert Wehn. "Beyond 100 gbit/s pipeline decoders for spatially coupled ldpc codes." *EURASIP Journal on Wireless Communications and Networking* 2022.1 (2022): 90.

Iyengar, Aravind R., et al. "Windowed decoding of protograph-based LDPC convolutional codes over erasure channels." *IEEE Transactions on Information Theory* 58.4 (2011): 2303-2320.

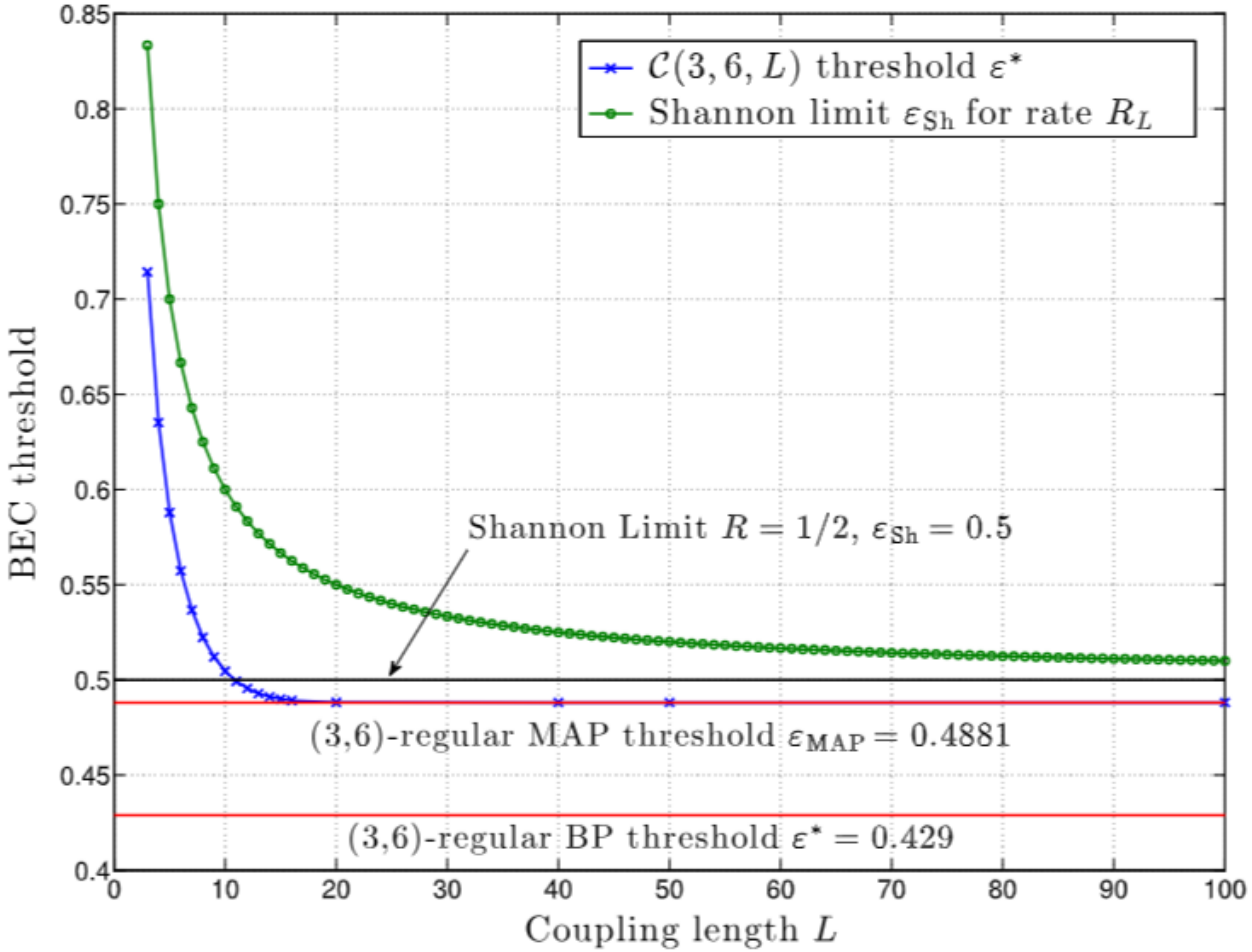


Fig. 10. BEC iterative BP decoding thresholds for $\mathcal{C}(3,6,L)$ SC-LDPC-BC ensembles with design rate $R_L = (L-2)/2L$ and the corresponding Shannon limit $\varepsilon_{\mathrm{Sh}} = 1 - R_L$ for rate R_L. Also shown are the BP and MAP decoding thresholds for the underlying (3,6)-regular LDPC-BC ensemble, $\varepsilon^* = 0.429$ and $\varepsilon_{\mathrm{MAP}} = 0.4881$, respectively, and the Shannon limit for $R = 1/2$ codes, $\varepsilon_{\mathrm{Sh}} = 0.5$.

Linear Minimum Distance of Protograph-Based LDPC Codes

Why Protograph-Based Code Ensembles?

- SC-LDPC code: Local structures for efficient decoding.
- Consider protograph-based code ensembles in order to both include **randomness** and preserve **local structures**.
    - Compared to the usual LDPC ensembles over all Tanner graphs with the same degree distribution.

The Minimum Distance of Protograph-Based Codes

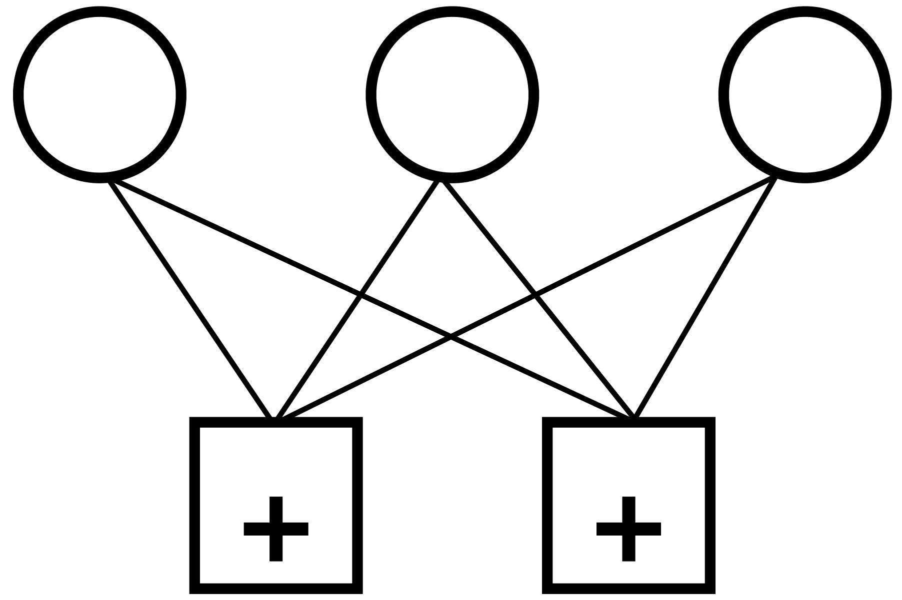
- The structures of protograph-based codes are random.
- Use probabilistic argument to characterize the average performance over the code ensemble.
- With high probability, $d_{\text{min}} > \delta_{\text{min}} n$ as lifting factor M goes to infinity.
  - d_{min}: Minimum distance of the code.
  - $n \triangleq M n_v$ is the codelength, where n_v is the number of variable nodes in the protograph.
  - δ_{min}: The **minimum distance growth rate** of a code.
    - A metric to compare different codes w.r.t. minimum distances.

Code Ensemble and the Underlying Probability

- Given a protograph.
- Define the **ensemble** of codes to be the set of all the Tanner graphs that can be constructed by M-lifting the protograph [1].
- Define $\mathbb{P}^{(M)}$ to be the uniform probability measure over the ensemble [3].

Example

- Consider the protograph $\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and the lifting factor $M = 3$.
- Tanner graph of $\mathbf{B}$:

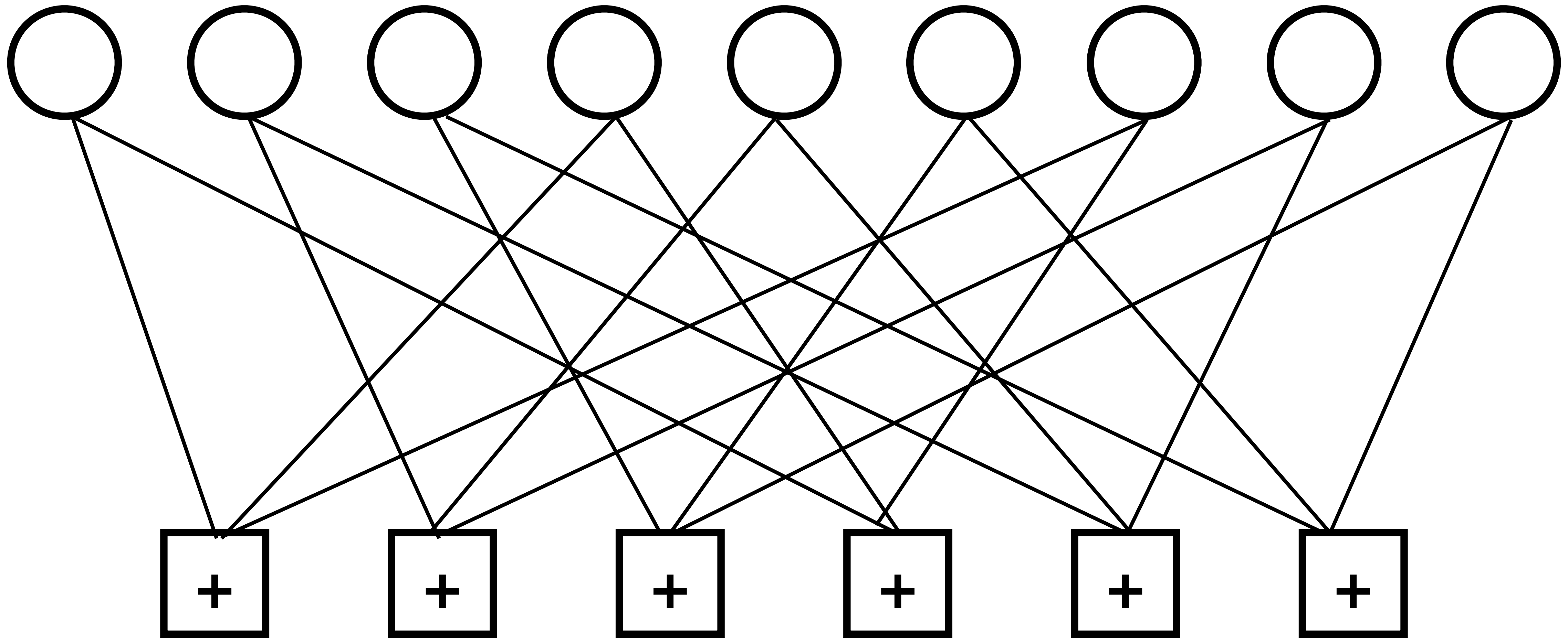


- Minimum distance of this protograph is 2.
- There are $(3!)^6$ protographs in this ensemble.
  - Each with probability $\frac{1}{(3!)^6}$.

Example (Conti'd)

- A Tanner graph in the 3-lifted ensemble:

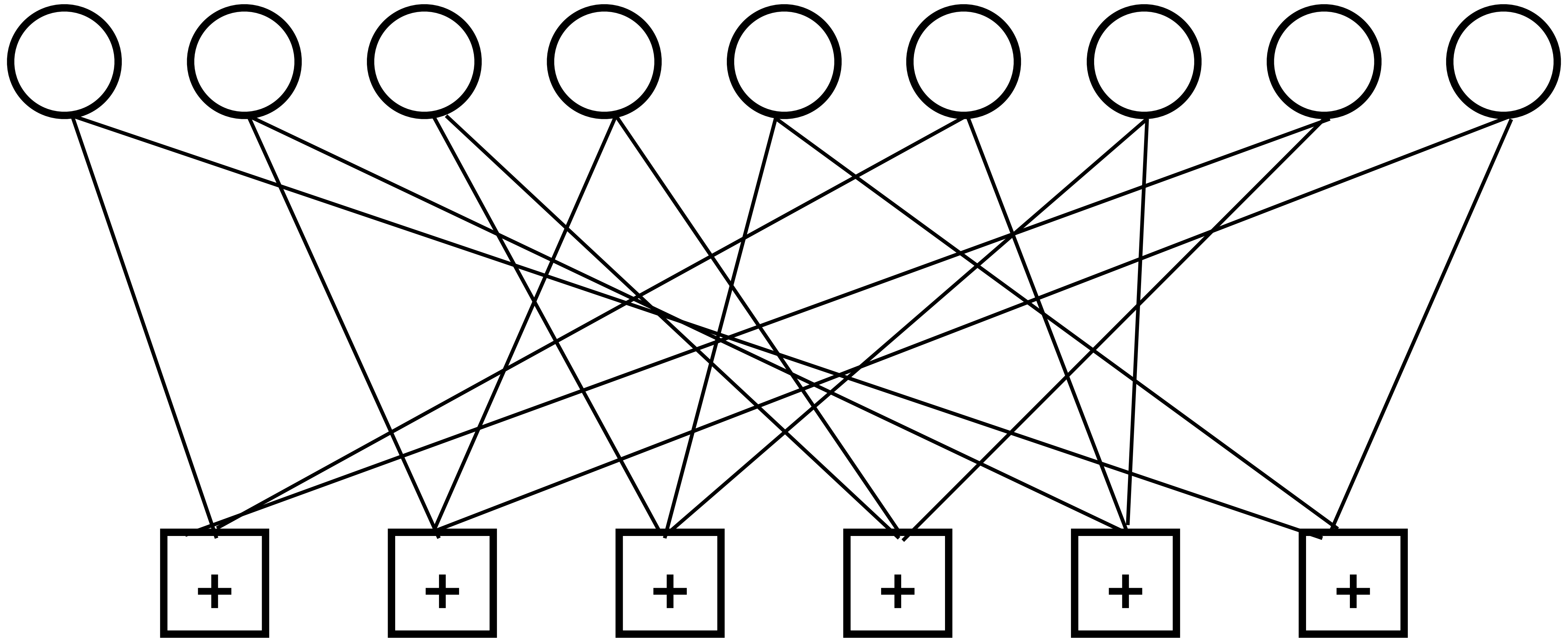
- $$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$



- The minimum distance of this graph is 2.
    - The same as that of the protograph.

Example (Conti'd)

- Another Tanner graph in the 3-lifted ensemble:



- $$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The minimum distance of this graph is 4.
    - Increased minimum distance (compared with the protograph).

Average Number of Codewords of a Specific Weight

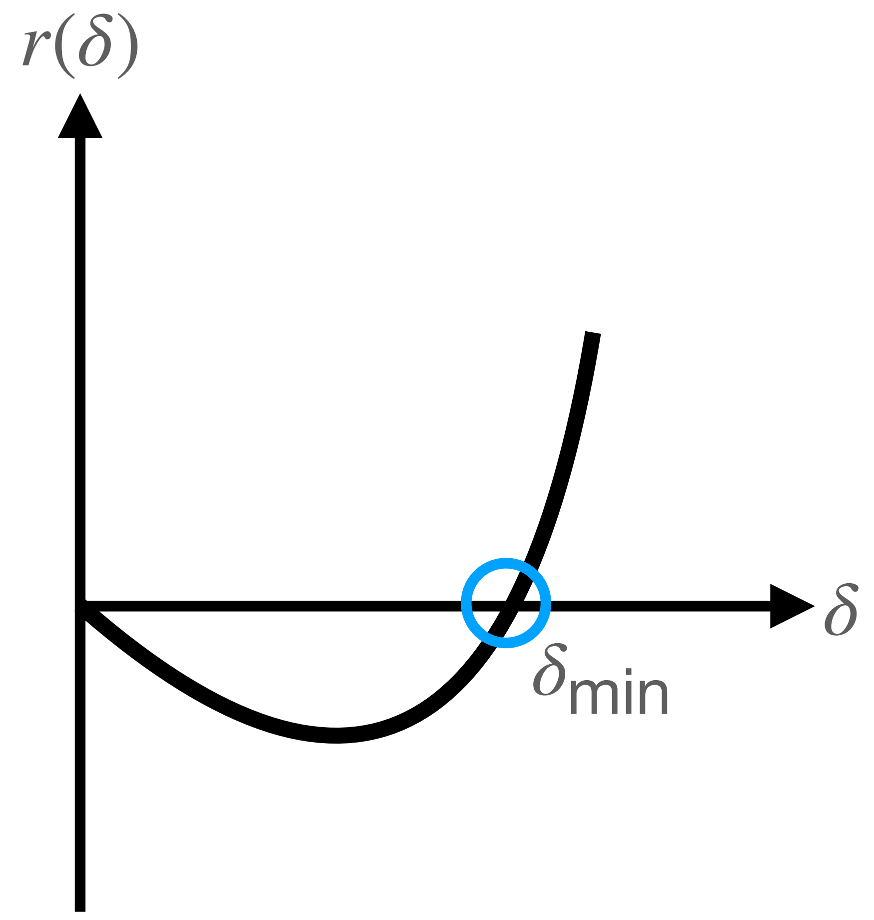
- For $1 \leq d \leq Mn_v$, define $A_d^{(M)}$ as the average number of codewords of weight d in the M-lifted code ensemble.
- More explicitly, $A_d^{(M)} \triangleq \mathbb{E}^{(M)}[X_d^{(M)}]$, where
  - $X_d^{(M)}$ is the number of codewords of weight d in **a** code from the M-lifted ensemble.
    - $X_d^{(M)}$ is a **random** variable;
  - $\mathbb{E}^{(M)}$ is the expectation operator w.r.t. the probability measure $\mathbb{P}^{(M)}$ [3].

Asymptotic Spectral Shape Function

- Define the **asymptotic spectral shape function** $r(\delta) \triangleq \limsup_{M \to \infty} r_M(\delta)$, where $r_M(\delta) \triangleq \frac{\ln(A_{\delta n}^{(M)})}{n}$.
  - Operational meaning of $r_M(\delta)$: The exponent of the average number of codewords of weight δn.
    - For example, say $\delta = 0.1$.
    - There are 2^n binary words of length n.
    - If there are (on average) $2^{0.3n}$ words that are codewords of weight $0.1n$, then $r_M(0.1) = 0.3$.
      - Up to some constant factor depending on the base of logarithm.
- $r(\delta)$ only depends on the protograph.

Minimum Distance Growth Rate

- Define the **minimum distance growth rate** δ_{min} to be the first zero-crossing of the function $r(\delta)$.
  - That is, $r(\delta_{\mathrm{min}}) = 0$ and $r(\delta) < 0$ for $0 < \delta < \delta_{\mathrm{min}}$.
  - If exists.
  - For a given protograph, one can calculate $r(\delta)$ by the recursive method in [2] and [5] to determine whether δ_{min} exists, and if exists, its value.



Probabilistic Guarantee of Linear Minimum Distance

- Consider the probability $\mathbb{P}^{(M)}(d_{\text{min}}^{(M)} < \delta_{\text{min}} n) \leq \sum_{d=1}^{\delta_{\text{min}} n-1} A_d^{(M)}$.
  - $d_{\text{min}}^{(M)}$ is the minimum distance of an M-lifted code in the ensemble, which is a **random** variable.
  - Can be derived by the union bound and Markov's inequality [4, Appendix B]:
    - $\mathbb{P}^{(M)}(d_{\text{min}}^{(M)} < \delta_{\text{min}} n) = \mathbb{P}^{(M)}(\text{There is a codeword of weight } < \delta_{\text{min}} n) = \mathbb{P}^{(M)}(\bigcup_{d=1}^{\delta_{\text{min}} n-1} \{X_d^{(M)} \geq 1\})$.
    - By the union bound, we have $\mathbb{P}^{(M)}(\bigcup_{d=1}^{\delta_{\text{min}} n-1} \{X_d \geq 1\}) \leq \sum_{d=1}^{\delta_{\text{min}} n-1} \mathbb{P}^{(M)}(X_d \geq 1)$.
    - By Markov's inequality, we have $\sum_{d=1}^{\delta_{\text{min}} n-1} \mathbb{P}^{(M)}(X_d \geq 1) \leq \sum_{d=1}^{\delta_{\text{min}} n-1} \frac{\mathbb{E}^{(M)}[X_d]}{1} = \sum_{d=1}^{\delta_{\text{min}} n-1} A_d^{(M)}$.

Probabilistic Guarantee of Linear Minimum Distance (Conti'd)

- We have $\sum_{d=1}^{\delta_{\text{min}} n-1} A_d^{(M)} \to 0$ as M goes to infinity.
  - Intuition: There are $\mathcal{O}(M)$ terms in the summation, each of which decays exponentially in M.
  - Rigorous proof: See, for example, [4, Appendix B].

Probabilistic Guarantee of Linear Minimum Distance (Conti'd)

- Combining the previous two results, we have

$$\mathbb{P}^{(M)}(d_{\text{min}}^{(M)} < \delta_{\text{min}} n) \leq \sum_{d=1}^{\delta_{\text{min}} n - 1} A_d^{(M)} \to 0.$$

- For large enough M, with high probability, the minimum distance of the M-lifted ensemble is **at least $\delta_{\text{min}} n$**.
  - Recall again that the codelength is $n = Mn_v$.
  - This result explains why δ_{min} is called the minimum distance growth rate.

Comments

- The analysis of the minimum distance of a code ensemble using $A_d^{(M)}$ and $r(\delta)$ can be traced back to the thesis of Gallager [6, Chapter II].
- In [7], exact expressions of $r(\delta)$ for several code ensembles are given.
- A similar approach can be applied for the asymptotic size of trapping sets [8].
- In addition to the recursive method of computing $r(\delta)$ in [2] and [5], one can also find $r(\delta)$ as the solution to an optimization problem [3, Theorem 1].
  - Derivation based on Sanov's theorem.
  - Sanov's theorem can also be used similarly in the analysis of trapping sets [8, Theorem 3.3].

Numerical Examples

C(J,K,L) SC-LDPC-BC Ensembles

- [1, Definition 6].
- Let J, K, L be positive integers.
  - L is the coupling length.
- The $\mathscr{C}(J, K, L)$ ensemble is the weight-lifting code ensemble whose protograph has a parity check matrix on the right [1, (8)].
  - $a = \text{gcd}(J, K)$ be the greatest common divisors of J and K.
  - Write $J = aJ'$ and $K = aK'$.
  - $w = a - 1$.
  - Each $\mathbf{B}_j$ is a J' by K' all-one matrix.
  - The vertical pattern is repeated L times.

$$\mathbf{B}_{[0,L-1]} = \begin{bmatrix} \mathbf{B}_0 & & & \\ \mathbf{B}_1 & \mathbf{B}_0 & & \\ \vdots & \mathbf{B}_1 & \ddots & \\ \mathbf{B}_w & \vdots & \ddots & \mathbf{B}_0 \\ & \mathbf{B}_w & & \mathbf{B}_1 \\ & & \ddots & \vdots \\ & & & \mathbf{B}_w \end{bmatrix}$$

Minimum Distance Growth Rate of SC-LDPC-BC Codes

- Consider the $\mathscr{C}(3{,}6{,}L)$ SC-LDPC-BC code ensembles [1, Definition 6].
- That is, the code ensemble with protograph

$$\mathbf{B}_{[0,L-1]} = \begin{bmatrix} 1\ 1 & & & \\ 1\ 1 & 1\ 1 & & \\ 1\ 1 & 1\ 1 & \ddots & \\ & 1\ 1 & \ddots & 1\ 1 \\ & & \ddots & 1\ 1 \\ & & & 1\ 1 \end{bmatrix}_{(L+2)\times 2L} .$$

- On the right are the Tanner graph of $\mathbf{B}_{[0,L-1]}$ [1, Fig. 6] and the minimum distance growth rate for different L [1, Table I].

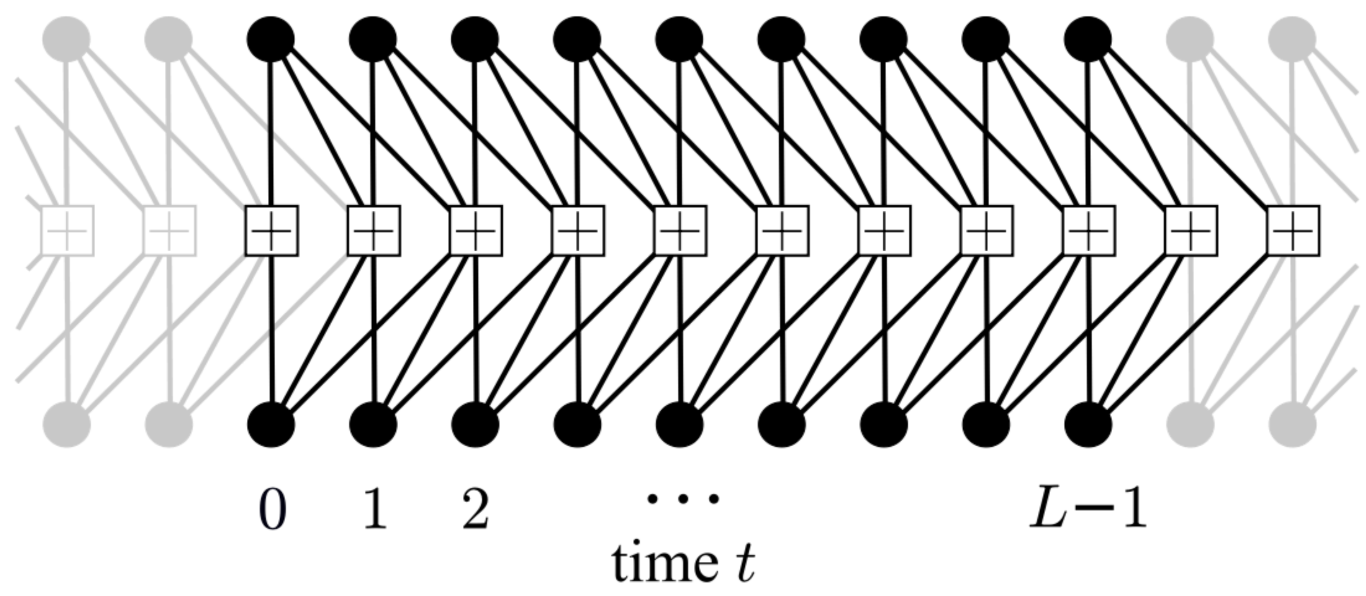


Fig. 6. Protograph of a SC-LDPC-BC ensemble (highlighted in black) with coupling length L and coupling width $w = 2$ obtained by terminating a (3, 6)-regular convolutional protograph.

TABLE I
MINIMUM DISTANCE GROWTH RATES FOR THE $\mathcal{C}(3, 6, L)$ SC-LDPC-BC ENSEMBLES

L	Design Rate R_L	Growth rate $\delta_{\min}^{(L)}$	$\delta_{\min}^{(L)} L/(w+1)$
3	1/6	0.1419	0.142
4	1/4	0.0814	0.109
5	3/10	0.0573	0.096
6	1/3	0.0449	0.090
7	5/14	0.0374	0.087
8	3/8	0.0324	0.086
9	7/18	0.0287	0.086
10	2/5	0.0258	0.086
20	9/20	0.0129	0.086
∞	1/2	0	

Minimum Distance Growth Rate of SC-LPDC-BC Codes (Conti'd)

- Consider $\mathscr{C}(J, 2J, L)$ code ensembles.
- On the right are the minimum distance growth rate v.s. the design rate of $\mathscr{C}(J, 2J, L)$ code ensembles [1, Fig. 9].
    - The Gilbert-Varshamov (G-V) bound: There exist a code with $R \geq 1 - H(\delta_{\text{min}})$ [9], [10], [11, Problem 1.15].
    - The gap between the proposed codes in [1] and the G-V bound is expected: It is difficult to explicitly construct a binary linear code achieving the G-V bound [12].

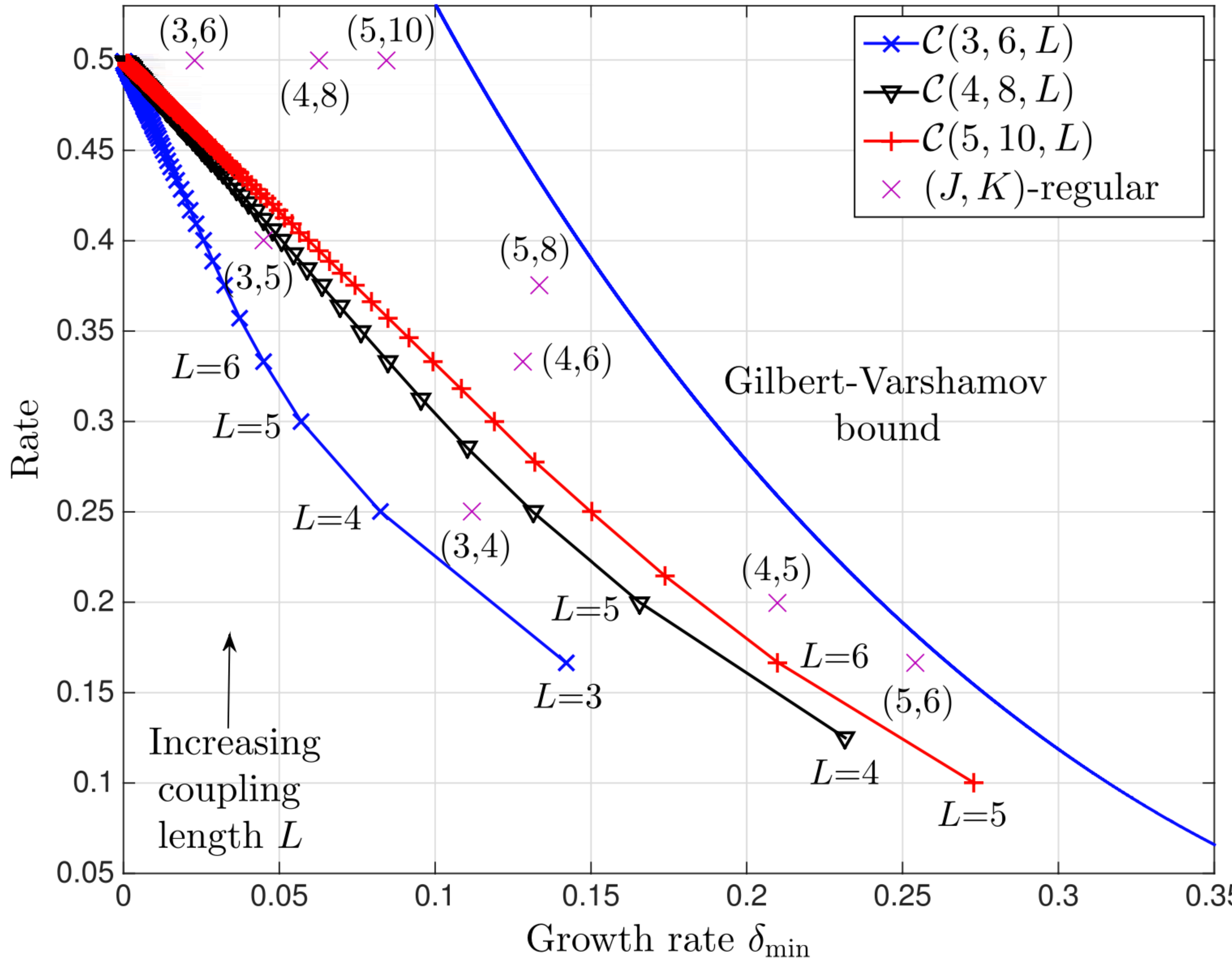


Fig. 9. Minimum distance growth rates for $\mathcal{C}(J, 2J, L)$ SC-LDPC-BC ensembles with design rate $R_L = (L - J + 1)/2L$ and some (J, K)-regular LDPC-BC ensembles with design rate $R = 1 - J/K$. Also shown is the Gilbert-Varshamov bound for random block code minimum distance growth rates.

BEC Threshold and Minimum Distance of SC-LDPC-BC Ensembles

- [1, Fig. 12].

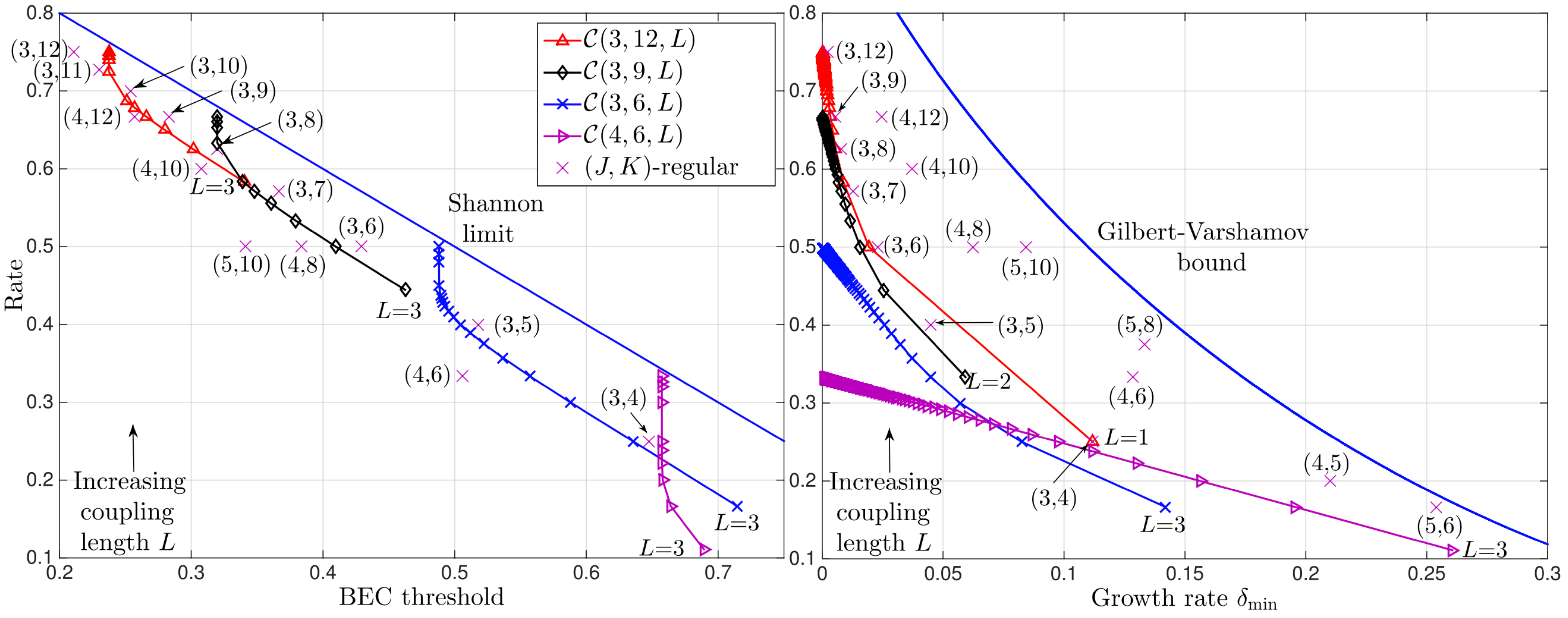


Fig. 12. BEC iterative BP decoding thresholds and minimum distance growth rates of four $\mathcal{C}(J,K,L)$ SC-LDPC-BC ensembles and several uncoupled (J,K)-regular LDPC-BC ensembles.

Conclusion

Conclusion

- SC-LDPC code ensembles constructed from protographs have the following property:
  - 1. BP threshold approaches MAP threshold.
  - 2. Minimum distance grows linearly in the codelength n.

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- **(Main reference)** [1] D. G. M. Mitchell, M. Lentmaier and D. J. Costello, "Spatially Coupled LDPC Codes Constructed From Protographs," in *IEEE Transactions on Information Theory*, vol. 61, no. 9, pp. 4866-4889, Sept. 2015, doi: 10.1109/TIT.2015.2453267.
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