Capacity of Noisy Permutation Channels

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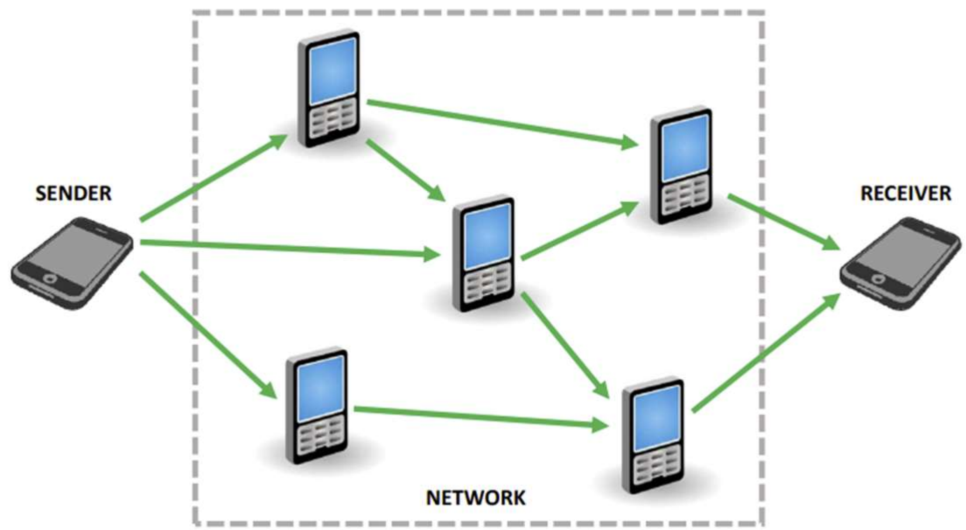
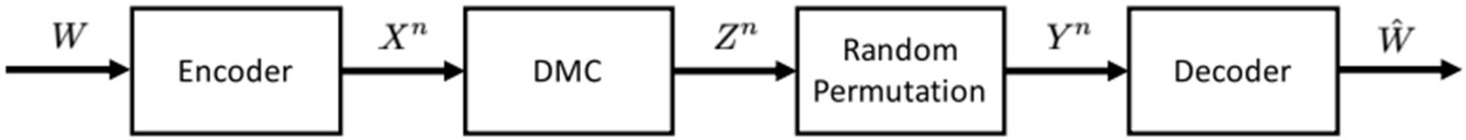
1. Motivation and Problem Statement

- Some channels do not preserve order (abc → cba)
  - DNA storage
  - Packets in a network

Note: while mentioned in literature, these applications are not very compelling



1. Motivation and Problem Statement



- A DMC with an extra random permutation step
  - Draw one of the many possible permutations of n symbols uniformly
- The order of the received symbols is uninformative
  - Instead of sequences into/out of a channel, we care about types
  - A type tells us everything about a sequence except the ordering

1. Motivation and Problem Statement

Definitions: Channel

- Message $W \in \{1,2,\ldots,M\}$
- Encoded into codeword $X^n = f_n(W)$ where $\mathrm{X} \in \mathcal{X} = \{1,2,\ldots,q\}$
- DMC channel input X^n yields output Z^n, $\mathrm{Z} \in \mathcal{Y} = \{1,2,\ldots,k\}$
- Random permutation yields $Y^n \in \mathcal{Y}^n$
- Decoded into output $\widehat{W} = g_n(Y^n)$
- $P_{\mathrm{error}}^{(n)} = \Pr\{W \neq \widehat{W}\}$

Rate is $R = \frac{\log M}{\log n}$ to account for loss of information from random permutation

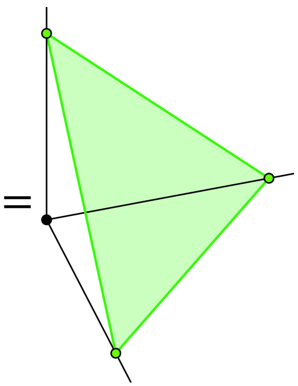
- The number of distinguishable **types** is polynomial in n
- The number of distinguishable **sequences** is exponential in n

1. Motivation and Problem Statement

Definitions (continued): General

- Probability simplex:

$$\Delta_{q-1} = \left\{ (\pi_1, \pi_2, \dots, \pi_n) : \sum_{i=1}^{q} \pi_i = \right.$$



- Method of Types

$$\mathcal{P}_n = \left\{ P \in \Delta_{k-1} : P = \left(\frac{a_1}{n}, \frac{a_2}{n}, \dots, \frac{a_2}{n} \right) \text{ where } a_1, a_2, \dots, a_n \in \{0, 1, \dots, n\} \right\}$$

$$T_n(P) = \left\{ x^n \in \mathcal{X}^n : P = \left(\frac{N(a_1|x)}{n}, \frac{N(a_2|x)}{n}, \dots, \frac{N(a_n|x)}{n} \right) \right\}$$

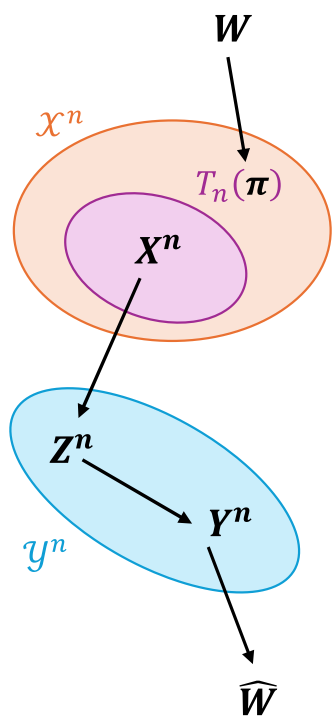
1. Motivation and Problem Statement

Definitions (continued): Channel Distributions

- $P_{Z|X}$ is a $q \times k$ DMC matrix
  - $(P_{Z|X})_{b,j} = P_{Z|X}(j|b) = p_{bj}$ for $b \in \mathcal{X}$ and $j \in \mathcal{Y}$
  - Rows sum to 1
  - $P_{Z|X}$ s.p. (strictly positive) when $p_{bj} > 0 \;\; \forall b \in \mathcal{X}, j \in \mathcal{Y}$

- $P_{Y^n|\pi} = P_{Y|X}^n \circ U$ is the distribution under fixed type $\pi \in \mathcal{P}_n$
  - Where we draw U uniformly from $T_n(\pi)$ (A)
  - Then pass independently through the DMC $P_{Y|X}$
  - Note that $P_{Z|X}$ and $P_{Y|X}$ are interchangeable

- Marginal P_Y for any $Y_t \in Y^n \sim P_{Y^n|\pi}$
  - Marginal is independent of index t due to (A)



1. Motivation and Problem Statement

Definitions (continued): Other Distributions:

- Q_Y is any distribution over Y
- $Q_Y^n(y^n) = \prod_{i=1}^n Q_Y\ (y_i)$ is the product distribution
- Categorical distribution $Q_{Y|\mu}$ with $\mu = (\mu_1, \ldots, \mu_k) \in \Delta_{k-1}$ with $Q_{Y|\mu}(j) = \mu_j$
- Multinomial $Q_{Y|\mu}^n$ has no direct relevance to a noisy permutation, but simplifies analysis

Nats are used, not bits

1. Motivation and Problem Statement

Theorem 1: Main result

For s.p. (strictly positive) $P_{Y|X}$, the capacity of the noisy permutation channel is

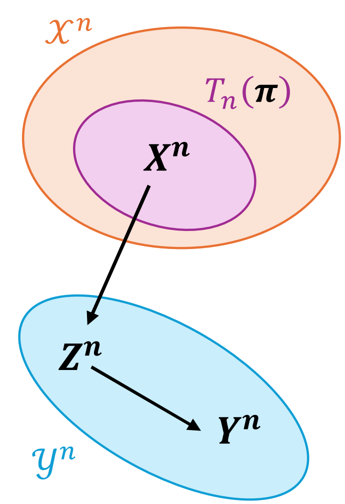
$$C_{\text{perm}}(P_{Z|X}) = \frac{\text{rank}(P_{Z|X}) - 1}{2}$$

From [Makur, 2020] we know that (for s.p. $P_{Y|X}$)

$$C_{\text{perm}}(P_{Z|X}) \geq \frac{\text{rank}(P_{Z|X}) - 1}{2}$$

Thus, we seek a converse (upper) bound to prove theorem 1

1. Motivation and Problem Statement



Markov chain is

$$\pi \to X^n \to Z^n \to Y^n$$

for some n-type $\pi \in \mathcal{P}_n$ chosen by the encoder. Thus, should find the upper bound of $I(\pi; Y^n)$ to get capacity.

Further, we know

$$I(\pi; Y^n) \leq \max_{\pi} D\left(P_{Y^n|\pi} \| Q_{Y^n}\right)$$

Proposition 1: Covering for Noisy Permutation Channel

For a noisy permutation channel with DMC $P_{Y|X}$ and any n-type $\pi \in \mathcal{P}_n$, we **assume** that

$$D\left(P_{Y^n|\pi} \| Q^n_{Y|\bar{\mu}}\right) \leq nD(P_Y \| Q_Y) + f(n)$$

for any distribution Q_{Y^n}. Then,

$$C_{\text{perm}}\left(P_{Z|X}\right) \leq \frac{\text{rank}(P_{Z|X}) - 1}{2} + \lim_{n\to\infty} \frac{f(n)}{\log n}$$

2. Covering Converse

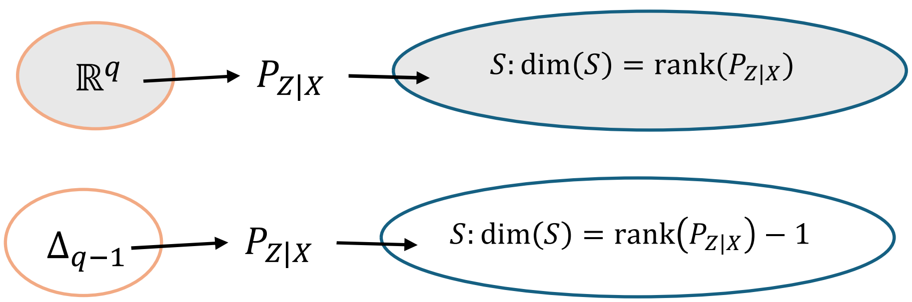
Covering a space of distributions

Consider the space of possible marginals $\boxed{P_Y}$

$$\mathcal{L}(P_{Y|X}) = \bigcup_{\pi \in \Delta_{k-1}} \boxed{\left(\sum_i \pi_i p_{i1}, \sum_i \pi_i p_{i2}, \ldots, \sum_i \pi_i p_{ik}\right)}$$

with dimension $l = \boxed{\mathrm{rank}(P_{Z|X})}\ \boxed{-1}$

DMC Probability vec



2. Covering Converse

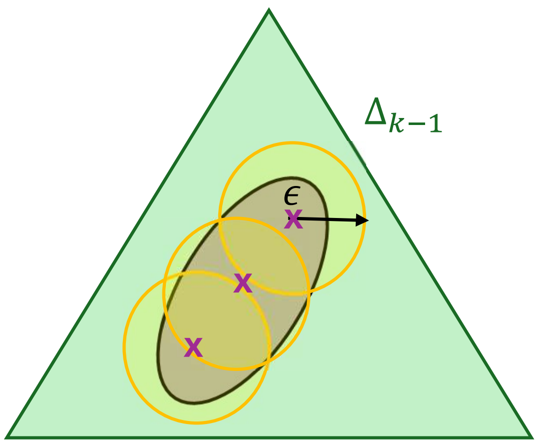
Background: ϵ-net covering are a popular tool in CS

Covering a space of distributions (continued)

Define the **ϵ-net covering** of $\mathcal{L}(P_{Y|X})$ as N_n (also, let $\epsilon = 1/n$)

N_n is a set of "covering centers" $\bar{\mu} = (\mu_1, \mu_2, \dots, \mu_k) \in \Delta_{k-1}$, and not necessarily $\bar{\mu} \in \mathcal{L}(P_{Y|X})$

Idea: use a few vectors to describe a space of vectors



1. Every point π in space $\mathcal{L}(P_{Y|X})$ is ϵ-close to some center $\bar{\mu}$, in terms of KLD

$$\max_{\pi \in \mathcal{L}(P_{Y|X})} \min_{\bar{\mu} \in N_n} D\left(Q_{Y|\mu} \parallel Q_{Y|\bar{\mu}}\right) \leq \epsilon = \frac{1}{n}$$

2. Covering Converse

Covering a space of distributions (continued)

2. Cardinality of covering has convenient form

$$|N_n| \leq C(q,l)(nl)^{l/2}$$ Theorem 4

- $C(q,l)$ is given in the text, but is independent of n and will be made to vanish
- Reminder: $\epsilon = 1/n$

Why bother with coverings?

$$I(\pi; Y^n) \leq \log|N_n| + \max_{\pi \in \mathcal{P}_n} \min_{\bar{\mu} \in N_n} D\left(P_{Y^n|\pi} \parallel Q^n_{Y|\bar{\mu}}\right)$$ [Yang 1999]

2. Covering Converse

$$I(\pi; Y^n) \leq \log|N_n| + \max_{\pi \in \mathcal{P}_n} \min_{\overline{\mu} \in N_n} D\left(P_{Y^n|\pi} \parallel Q^n_{Y|\overline{\mu}}\right)$$ [Yang 1999]

$$\leq \log|N_n| + \max_{\pi \in \mathcal{P}_n} \min_{\overline{\mu} \in N_n} nD(P_Y \| Q_Y) + f(n)$$ Assume (for now)

$$\leq \log|N_n| + f(n) + \max_{\pi \in \mathcal{P}_n} \min_{\overline{\mu} \in N_n} nD(P_Y \| Q_Y)$$

$$\leq \log|N_n| + f(n) + n\frac{1}{n}$$ By covering

$$\leq \log\left(C(q,l)(nl)^{l/2}\right) + f(n) + n\frac{1}{n}$$ By covering

$$\leq \log C(q,l) + \frac{l}{2}(\log l + \log n) + f(n)) + n\frac{1}{n}$$

$$\leq \frac{l}{2}\log n + c' + f(n)$$

c' contains all terms not depending on n

2. Covering Converse

Proof of proposition 1

$$\log M \leq I(\pi; Y^n) \leq \frac{l}{2}\log n + c' + f(n)$$

By $R = \frac{\log M}{\log}$ we have

$$R \leq \frac{l}{2} + \frac{c'}{\log n} + \frac{f(n)}{\log n}$$
$$\lim_{n\to\infty} R = \frac{l}{2} + \lim_{n\to\infty} \frac{f(n)}{\log n}$$

Thus

$$C_{\text{perm}}(P_{Z|X}) \leq \frac{\text{rank}(P_{Z|X}) - 1}{2} + \lim_{n\to\infty} \frac{f(n)}{\log n}$$ QED (Proposition 1)

2. Covering Converse

Proposition 1: Covering for Noisy Permutation Channel

For a noisy permutation channel with DMC $P_{Y|X}$ and any n-type $\pi \in \mathcal{P}_n$, we **assume** that

$$D\left(P_{Y^n|\pi} \middle\| Q^n_{Y|\bar{\mu}}\right) \leq nD(P_Y \| Q_Y) + f(n)$$

Next (Theorem 2)

For any distribution $\tilde{Q}_{Y^n}$. Then,

$$C_{\text{perm}}\left(P_{Z|X}\right) \leq \frac{\text{rank}(P_{Z|X}) - 1}{2} + \lim_{n\to\infty} \frac{f(n)}{\log n}$$

Theorem 1: Main result

For s.p. (strictly positive) $P_{Y|X}$, the capacity of the noisy permutation channel is

$$C_{\text{perm}}\left(P_{Z|X}\right) = \frac{\text{rank}(P_{Z|X}) - 1}{2}$$

3. Divergence under Fixed Types

- Recap **Theorem 2**: When $P_{Y|X}$ is s.p. (stricyly positive), $\exists\ c = c(P_{Y|X})$, $\forall P \in \mathcal{P}_n$, with $U \sim T_n(P)$ in uniform, $\forall Q_Y$:

$$nD(P_Y \| Q_Y) \le D(P_{Y|X}^n \circ U \| Q_Y^n) \le nD(P_Y \| Q_Y) + c$$

Final goal

3. Divergence under Fixed Types

- **Proposition 3**: Consider $(X,Y)^n \sim (P \times P_{Y|X})$ in i.i.d, with $A = 1\{X^n \in T_n(P)\}$:

$$D\left(P_{Y|X}^n \circ U \,\|\, Q_Y^n\right) = nD\left(P_Y \,\|\, Q_Y\right) + \underbrace{\sum_{y^n \in Y^n} P\left[Y^n = y^n \mid A=1\right] \cdot \log \frac{P\left[A=1 \mid Y^n = y^n\right]}{P(A=1)}}_{②}$$

- **Proof (1/2)**:
  - with: $(P_{Y|X}^n \circ U)(y^n) = P[Y^n = y^n | A = 1]$;
  - $D\left(P_{Y|X}^n \circ U \,\|\, Q_Y^n\right)$

$$= \sum_{y^n} P\left[Y^n = y^n \mid A=1\right] \cdot \log \frac{P\left[Y^n = y^n \mid A=1\right]}{Q_Y^n(y^n)}$$ ---------------- KL-Divergence

$$= \sum_{y^n} P\left[Y^n = y^n \mid A=1\right] \cdot \log \frac{P\left[A=1 \mid Y^n = y^n\right] P(Y^n = y^n)}{P(A=1) Q_Y^n(y^n)}$$ ---------------- Baye's Theorem

$$= \underbrace{\mathrm{E}\left[\log \frac{P_Y^n(Y^n)}{Q_Y^n(Y^n)} \mid A=1\right]}_{①} + \underbrace{\mathrm{E}\left[\log \frac{P\left[\tilde{A}=1 \mid \tilde{Y}^n = Y^n\right]}{P(\tilde{A}=1)} \mid A=1\right]}_{②}$$

3. Divergence under Fixed Types

- **Proposition 3**: Consider $(X,Y)^n \sim (P \times P_{Y|X})$ in i.i.d, with $A = 1\{X^n \in T_n(P)\}$:

$$D\left(P_{Y|X}^n \circ U \,\|\, Q_Y^n\right) = nD\left(P_Y \,\|\, Q_Y\right) + \sum_{y^n \in Y^n} P\left[Y^n = y^n \mid A=1\right] \cdot \log \frac{P\left[A=1 \mid Y^n = y^n\right]}{P(A=1)}$$

①　②

- **Proof (2/2)**:

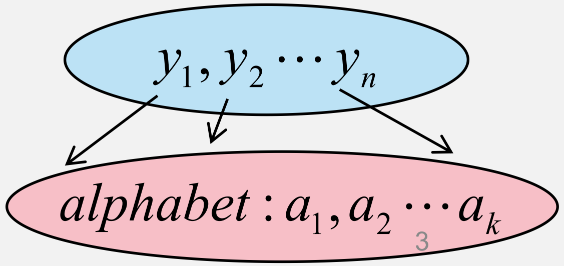
$$\mathrm{E}\left[\log \frac{P_Y^n(Y^n)}{Q_Y^n(Y^n)} \mid A=1\right]$$

$$= \sum_{y^n} P[Y^n = y^n \mid A=1] \log \frac{P_Y^n(Y^n)}{Q_Y^n(Y^n)}$$

Expectation

$$= \sum_{y^n} P[Y^n = y^n \mid A=1] \sum_a n \frac{|\{t : y_t = a\}|}{n} \log \frac{P_Y(a)}{Q_Y(a)}$$

$$\log \frac{P_Y^n(Y^n)}{Q_Y^n(Y^n)} \overset{iid}{=} \sum_a |\{t : y_t = a\}| \log \frac{P_Y(a)}{Q_Y(a)}$$



$$= n \sum_a P_Y(a) \log \frac{P_Y(a)}{Q_Y(a)}$$

$$= nD(P_Y \,\|\, Q_Y)$$

3. Divergence under Fixed Types

- Recap **Theorem 2**: When $P_{Y|X}$ is s.p. (stricyly positive), $\exists\, c = c(P_{Y|X})$, $\forall P \in \mathcal{P}_n$, with $U \sim T_n(P)$ in uniform, $\forall Q_Y$:

$$nD(P_Y \| Q_Y) \le D(P_{Y|X}^n \circ U \| Q_Y^n) \le nD(P_Y \| Q_Y) + c$$

Final goal

- Recap **Proposition 3**: Consider a noisy permutation channel with DMC $P_{Y|X}$, for any $\pi \in \mathcal{P}_n$:

$$D(P_{Y|X}^n \circ U \| Q_Y^n) = nD(P_Y \| Q_Y) + \underbrace{\sum_{y^n \in Y^n} P[Y^n = y^n \mid A = 1] \cdot \log \frac{P[A = 1 \mid Y^n = y^n]}{P(A = 1)}}_{②}$$

- What's next?
- Equals to proof:

$$\boxed{0 \le \sum_{y^n \in Y^n} P[Y^n = y^n \mid A = 1] \cdot \log \frac{P[A = 1 \mid Y^n = y^n]}{P(A = 1)} \le c}$$

Final goal

3. Divergence under Fixed Types

- Equalized **Theorem 2**: When $P_{Y|X}$ is s.p. (stricyly positive), $\exists\ c = c(P_{Y|X})$, $\forall P \in \mathcal{P}_n$, with $U \sim T_n(P)$ in uniform, $\forall Q_Y$:

$$0 \le \sum_{y^n \in Y^n} P\left[Y^n = y^n \mid A = 1\right] \cdot \log \frac{P\left[A = 1 \mid Y^n = y^n\right]}{P(A=1)} \le c$$

Final goal

- Let's **prove** the lower bound:

$$\mathrm{E}\left[\log \frac{P\left[\tilde{A} = 1 \mid \tilde{Y}^n = y^n\right]}{P(\tilde{A}=1)} \mid A = 1\right]$$

$$= \sum_{y^n} P\left[Y^n = y^n \mid A = 1\right] \cdot \log \frac{P\left[A = 1 \mid Y^n = y^n\right]}{P(A=1)}$$

$$= \sum_{y^n} P\left[Y^n = y^n \mid A = 1\right] \cdot \log \frac{P\left[Y^n = y^n \mid A = 1\right]}{P\left[Y^n = y^n\right]}$$

Baye's Theorem

$$= D\left(P\left[Y^n \mid A = 1\right] \| P\left[Y^n\right]\right) \ge 0$$

KL-Divergence

- We only need to prove the upper bound for remaining part.

3. Divergence under Fixed Types

- Equalized **Theorem 2**: When $P_{Y|X}$ is s.p. (stricyly positive), $\exists\ c = c(P_{Y|X})$, $\forall P \in \mathcal{P}_n$, with $U \sim T_n(P)$ in uniform, $\forall Q_Y$:

$$0 \le \sum_{y^n \in Y^n} P\left[Y^n = y^n \mid A = 1\right] \cdot \log \frac{P\left[A = 1 \mid Y^n = y^n\right]}{P(A = 1)} \le c$$

Final goal

- The lower bound has been proved in previous slides:
- To prove the upper bound:

$$\mathrm{E}\left[\log \frac{P\left[\tilde{A} = 1 \mid \tilde{Y}^n = y^n\right]}{P(\tilde{A} = 1)} \mid A = 1\right] = \underbrace{\mathrm{E}\left[\log P\left[\tilde{A} = 1 \mid \tilde{Y}^n = y^n\right] \mid \tilde{A} = 1\right]}_{①} - \underbrace{\log P\left[A = 1\right]}_{②}$$

3. Divergence under Fixed Types

- **Lemma 2**: Let $(X,Y)^n \sim (P \times P_{Y|X})$ in i.i.d, with $A = 1\{X^n \in T_n(P)\}$, and $P = (p_1, \cdots p_q) \in \mathcal{P}_n$:

$$\log \frac{1}{P(A=1)} \le -\frac{1}{2}\log n + \sum_{i:p_i>0} \frac{1}{2}\log p_i n + \frac{q-1}{2}\log 2\pi + \frac{1}{12n}$$

②

- **Proof (1/1)**:
  - Stirling approximation: $\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \le n! \le \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$;

$$-\log P(A=1)$$

$$= -\log\left[\binom{n}{p_1 n, \cdots p_q n}\prod_{i=1}^{q} p_i^{p_i n}\right]$$ -------------------- Method of Types

$$= -\log\left(\frac{n!}{n^n}\right) - \log\left(\prod_{i=1}^{q}\frac{(p_i n)^{p_i n}}{(p_i n)!}\right)$$

$$\le n - \frac{1}{2}\log n - \frac{1}{2}\log 2\pi + \sum_{i=1}^{q}\left(-p_i n + \frac{1}{2}\log p_i n + \frac{1}{2}\log 2\pi + \frac{q}{12n}\right)$$ -------------------- Stirling approx.

3. Divergence under Fixed Types

- Recap equalized **Theorem 2**: When $P_{Y|X}$ is s.p. (stricyly positive), $\exists\ c = c(P_{Y|X})$, $\forall P \in \mathcal{P}_n$, with $U \sim T_n(P)$ in uniform, $\forall Q_Y$:

$$0 \le \sum_{y^n \in Y^n} P\left[Y^n = y^n \mid A = 1\right] \cdot \log \frac{P\left[A = 1 \mid Y^n = y^n\right]}{P(A = 1)} \le c$$

Final goal

- From Lemma 2 which we have proved:

$$\log \frac{1}{P(A=1)} \le -\frac{1}{2}\log n + \sum_{i:p_i>0} \frac{1}{2}\log p_i n + \frac{q-1}{2}\log 2\pi + \frac{1}{12n}$$

- We can get:

$$c = \mathrm{E}\left[\log \frac{P\left[\tilde{A} = 1 \mid \tilde{Y}^n = Y^n\right]}{P(\tilde{A} = 1)} \mid A = 1\right]$$

$$\le \frac{q-1}{2}\log n + c' + \mathrm{E}\left[\log P\left[\tilde{A} = 1 \mid \tilde{Y}^n = Y^n\right] \mid A = 1\right]$$

$$\le \frac{q-1}{2}\log n + c'$$

②

3. Divergence under Fixed Types

- **Theorem 5 (Petrov 2012)**: When $P(W_i - a_i \leq -\lambda_i/2) \geq b_i$, $P(W_i - a_i \geq \lambda_i/2) \geq b_i$, $i = 1, \cdots, n$, $\exists \alpha$:

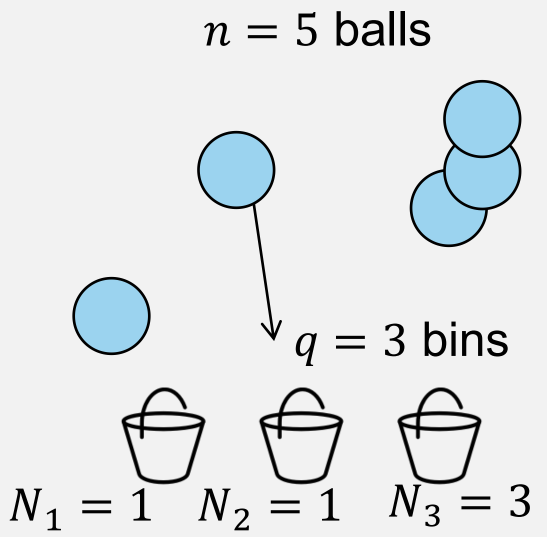
$$Q(S_n;\lambda) = \sup_z P[z \leq S_n \leq z+\lambda] \leq \alpha\lambda\left(\sum_{i=1}^{n} \lambda_i^2 b_i\right)^{-1/2}$$

- This provides us a conclusion: Let $W_i \sim Bernoulli(p_i)$, $S_n = \sum_{i=1}^{n} W_i$,

$$P[S_n = z] \leq Q(S_n; 1/2) \leq \frac{\alpha}{\sqrt{\sum_{i=1}^{n} \min\{p_i, 1-p_i\}}}$$

- **Lemma 3**: Independently throw n balls into one of q bins, we get:

$$P[N_1 = n\pi_1, \cdots, N_q = n\pi_q] \leq \frac{\alpha^{q-1}}{n^{(q-1)/2}\sqrt{B}}$$



$$B = c_*^{q-1} \frac{\Pi_b \pi_b}{\pi_{\max}} \quad c_* = \min_i \frac{c_-(i)}{c_+(i)} = \min_i \frac{\min_b\{p_{i,b}/\pi_b\}}{\max_b\{p_{i,b}/\pi_b\}}$$

$p_{i,b}$: Probability of i-th ball threw into b-th bin.

Increasing order

3. Divergence under Fixed Types

- **A useful bound**:

$$\min\left\{\frac{p_{i,b}}{\sum_{a=b}^{q} p_{i,a}}, 1-\frac{p_{i,b}}{\sum_{a=b}^{q} p_{i,a}}\right\}$$

$$=\min\left\{\frac{\pi_b \frac{p_{i,b}}{\pi_b}}{\sum_{a=b}^{q} \pi_a \frac{p_{i,a}}{\pi_a}}, \frac{\sum_{a>b}^{q} \pi_a \frac{p_{i,a}}{\pi_a}}{\sum_{a=b}^{q} \pi_a \frac{p_{i,a}}{\pi_a}}\right\}$$

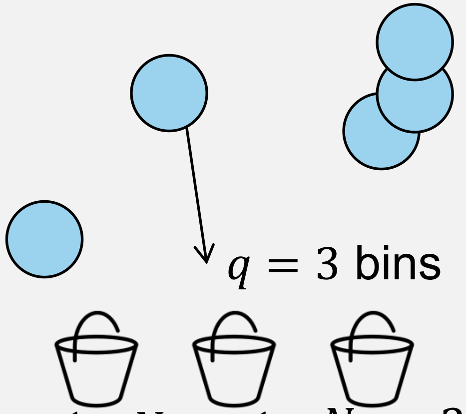
$$\geq \frac{\min_a \left\{\frac{p_{i,a}}{\pi_a}\right\}}{\max_a \left\{\frac{p_{i,a}}{\pi_a}\right\}} \min\left\{\frac{\pi_b}{\sum_{a=b}^{q} \pi_a}, \frac{\sum_{a>b}^{q} \pi_a}{\sum_{a=b}^{q} \pi_a}\right\}$$

$$\geq c_* \frac{\pi_b}{\sum_{a=b}^{q} \pi_a}$$

Increasing order: $\pi_b \leq \pi_{b+1} \leq \cdots$

$$c_* = \min_i \frac{c_-(i)}{c_+(i)} = \min_i \frac{\min_b \{p_{i,b} / \pi_b\}}{\max_b \{p_{i,b} / \pi_b\}}$$

$n = 5$ balls



$q = 3$ bins

$N_1 = 1$ $N_2 = 1$ $N_3 = 3$

$p_{i,b}$: Probability of i-th ball threw into b-th bin.

Increasing order

$$B = c_*^{q-1} \frac{\Pi_b \pi_b}{\pi_{\max}} \quad c_* = \min_i \frac{c_-(i)}{c_+(i)} = \min_i \frac{\min_b \{p_{i,b} / \pi_b\}}{\max_b \{p_{i,b} / \pi_b\}}$$

3. Divergence under Fixed Types

- **Lemma 3**: Independently throw n balls into one of q bins, we get:

$$P[N_1 = n\pi_1, \cdots, N_q = n\pi_q] \le \frac{\alpha^{q-1}}{n^{(q-1)/2}\sqrt{B}}$$

- **Proof**: $P[N_1 = n\pi_1, \cdots, N_q = n\pi_q]$

$$= \prod_{b=1}^{q} P[N_b = n\pi_b \mid N_1 = n\pi_1, \cdots, N_{b-1} = n\pi_{b-1}]$$

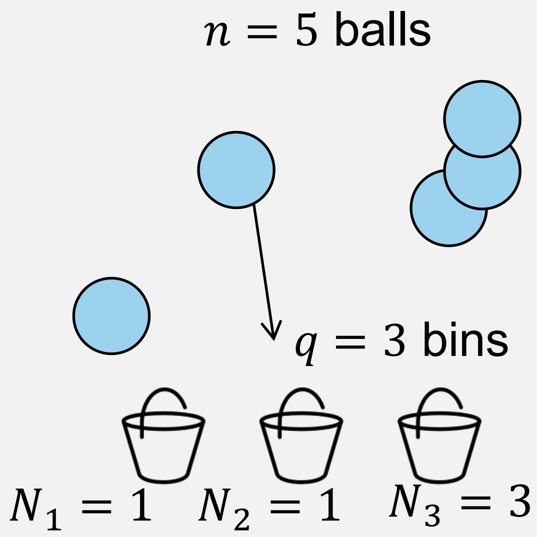
$$\le \prod_{b=1}^{q} \frac{\alpha}{\sqrt{\sum_{a=b}^{q} \min\{p_{i,a}, 1 - p_{ia}\}}}$$

[Petrov 2012]

$$\le \prod_{b=1}^{q} \frac{\alpha}{\sqrt{(n - \sum_{a=1}^{b-1} n\pi_a) c_* \frac{\pi_b}{\sum_{a=b}^{q} \pi_a}}}$$

useful bound

$$= \prod_{b=1}^{q} \frac{\alpha}{\sqrt{nc_*\pi_b}} = \frac{\alpha^{q-1}}{n^{(q-1)/2}\sqrt{B}}$$



$p_{i,b}$: Probability of i-th ball threw into b-th bin.

Increasing order

3. Divergence under Fixed Types

- To prove the upper bound:

①

$$\mathrm{E}\left[\log P\left[\tilde{A}=1 \mid \tilde{Y}^n = y^n\right] \mid \tilde{A}=1\right]$$
$$= \log P\left[N_1 = n\pi_1, \cdots, N_{q'} = n\pi_{q'}\right]$$
$$\le \log \frac{\alpha^{q'-1}}{n^{(q'-1)/2}\sqrt{B}}$$ ---------- Lemma3
$$= \log\left(\frac{\alpha^{q'-1}}{c_*^{(q'-1)/2}}\sqrt{\frac{n\pi_{\max}}{\Pi_{b=1}^{q'} n\pi_b}}\right)$$
$$= \frac{1}{2}\log n\pi_{\max} - \sum_{b:\pi_b>0}\frac{1}{2}\log n\pi_b + (q'-1)\log\frac{\alpha}{\sqrt{c_*}}$$ ---------- $\alpha \equiv const$, q' is number of symbols.
$$\le \frac{1}{2}\log n - \sum_{b:\pi_b>0}\frac{1}{2}\log n\pi_b + c'$$

$$c_* = \min_i \frac{c_-(i)}{c_+(i)} = \min_i \frac{\min_b\{p_{i,b}/\pi_b\}}{\max_b\{p_{i,b}/\pi_b\}}$$

3. Divergence under Fixed Types

- Recap **Theorem 2**: When $P_{Y|X}$ is s.p. (stricyly positive), $\exists\ c = c(P_{Y|X})$, $\forall P \in \mathcal{P}_n$, with $U \sim T_n(P)$ in uniform, $\forall Q_Y$:

$$nD(P_Y \| Q_Y) \le D(P_{Y|X}^n \circ U \| Q_Y^n) \le nD(P_Y \| Q_Y) + c$$

Proved

- Recap **Proposition 1**: Consider a noisy permutation channel with DMC $P_{Y|X}$, for any $\pi \in \mathcal{P}_n$, with $D(P_{Y|X}^n \circ U || Q_Y^n) \le nD(P_Y || Q_Y) + f(n)$, we get:

$$C_{perm}(P_{Y|X}) \le \frac{rank(P_{Y|X}) - 1}{2} + \lim_{n\to\infty} \frac{f(n)}{\log n}$$

Proved

- Recap [Makur 2020], that:

$$C_{perm}(P_{Y|X}) \ge \frac{rank(P_{Y|X}) - 1}{2}$$

Reference

- **Combine** them, we get the main result **Theorem 1**:

$$C_{perm}(P_{Y|X}) = \frac{rank(P_{Y|X}) - 1}{2}$$

Thank you for your attention!