Linear Algebra

#3

Question: How many sol^n's possible to $A\bar{x} = \bar{b}$, and what dim. is $N(A)$?

(a)
$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \text{Symm.} \\ \text{matrix} \end{bmatrix} \end{array}$$

Rank $= m = n$
Full rank

(b)
$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \text{Thin} \\ \text{matrix} \end{bmatrix} \end{array}$$

Rank $= n < m$
Full col. rank

(c)
$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \text{fat} \\ \text{matrix} \end{bmatrix} \end{array}$$

Rank $= m < n$
Full row rank

To see this, turn this matrix to a

(d)
$$\begin{bmatrix} \text{Matrix} \end{bmatrix}$$

Rank $< m$, Rank $< n$
Rank deficient matrix

- **Basis** : Linearly Independent

- **Dimensions** = $= |\text{Basis}|$

  ↳ A space can have

  Q: $\dim(C(A)) = ?$

- **Orthogonal vectors** :

  Q: wall and floor orthogonal ?

- **Norm** : $\begin{array}{cc} & \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \\ m & \end{array} \longrightarrow$ scalar $\Rightarrow \mathbb{R}^m \rightarrow \mathbb{R}$

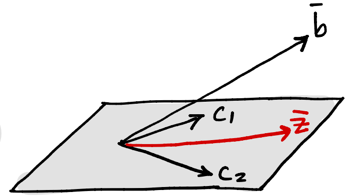
- **Length** = L_2 norm =

- **L_0 norm** = # of

- **uv^T** =

- **Symmetric matrix** ≡
- **$(AB)^{-1}$** = and $(AB)^T =$
- **A^TA** is a

Q: But say you have to solve $A\bar{x} = \bar{b}$ even though $\bar{b} \notin C(A)$.
What's the best you can do?

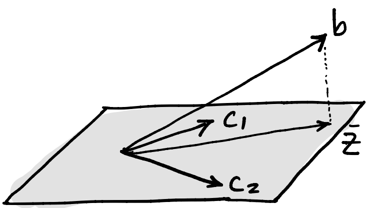


Ans: Produce a vector as $\bar{z} = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

s.t. $\bar{z}$ is closest possible to $\bar{b}$.

Now declare $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the approx sol^n to $A\bar{x} = \bar{b}$

What is "closest possible"?



Define closest possible as

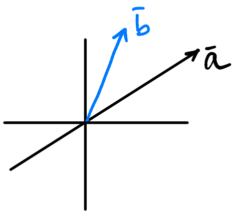
$\Rightarrow$ or

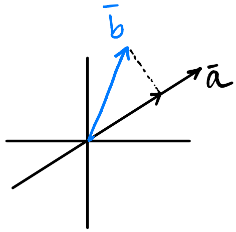
i.e., $\bar{z}$ is

Given this, find solution to $\bar{x}$ which is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Least Squares Solution

(1D case w.l.o.g.)





$\bar{z} = x \cdot \bar{a}$

$\bar{z} + \bar{e} =$

$\bar{e} =$

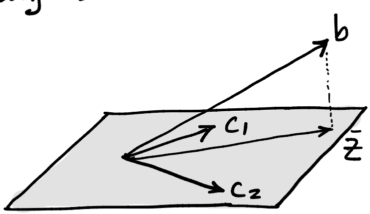
Now, since $\bar{a} \perp \bar{e}$,

or $\quad a^T(\qquad) = 0 \quad$ or

$\therefore \quad x =$

Now, take this to higher dimensions

say 2D.



$\bar{x} =$

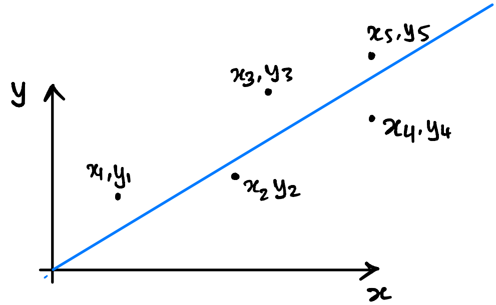
Projected vector:

$\bar{z} = A\bar{x} =$

$\therefore$ Projection matrix (P) =

Matrix P projects any vector to the

$\Rightarrow$ **Example : Regression** $\Rightarrow$ Which line $\boxed{y = mx + c}$ satifies all points.

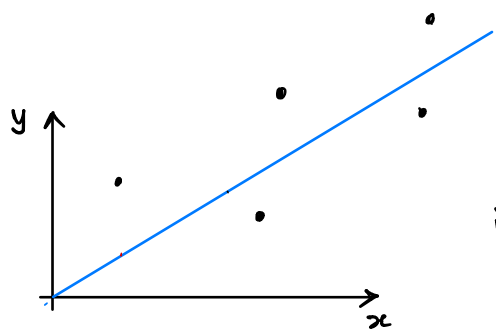


$$y_1 = m x_1 + c$$
$$y_2 = m x_2 + c$$
$$\vdots$$
$$y_n = m x_n + c$$

$\Rightarrow$

Since no solution exists $\Rightarrow$

so minimize error $e_i =$



For this minimization, we know solution $\bar{x}$

i.e., $\bar{x} =$

$$\begin{bmatrix} m \\ c \end{bmatrix} =$$

Eigenvalues and Eigenvectors.

$A\bar{x} = b$ or $\boxed{A(\cdot)}$

function or operation → A; input → $\bar{x}$; output → b

$A\bar{x} = \lambda\bar{x}$

} input vector only changes in mag. but not in direction ...

$x \equiv$

$\lambda \equiv$

How to obtain eigenvectors and eigenvalues?

$$Ax = \lambda x \Rightarrow A\bar{x} - \lambda\bar{x} = 0 \Rightarrow \qquad , x \neq 0$$

e.g.,

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad \therefore (A - \lambda I) = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$\text{Det}(A - \lambda I) = (3-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow (\lambda - 4)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 4, \quad \lambda_2 = 2$$

Now,

$$\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigen Decomposition and Diagonalization

$$\left.\begin{aligned} A x_1 &= \lambda_1 x_1 \\ A x_2 &= \lambda_2 x_2 \\ &\vdots \\ A x_n &= \lambda_n x_n \end{aligned}\right\}$$

$$\therefore \quad A S = S \Lambda$$

$\Rightarrow \quad A =$ $\rightarrow$ called

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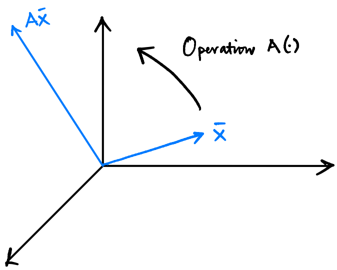
Intuition for $A = S \Lambda S^{-1}$

1



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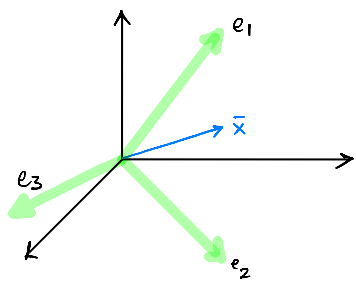

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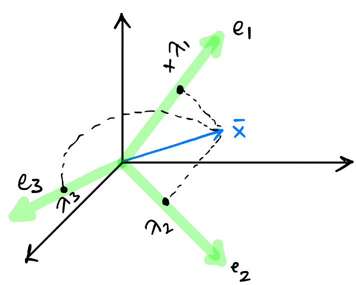

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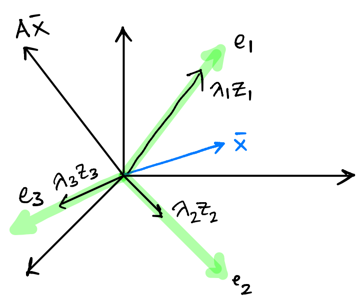

So what is A^2 ? A^3 ? A^n ?