DSP : Discrete Fourier Transform (DFT)

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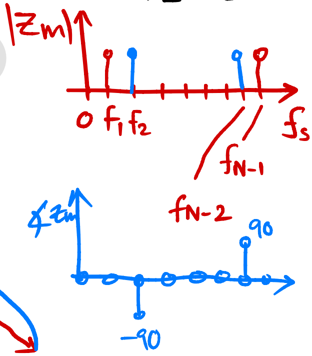
→ 3 important properties of DFT

① DFT is linear

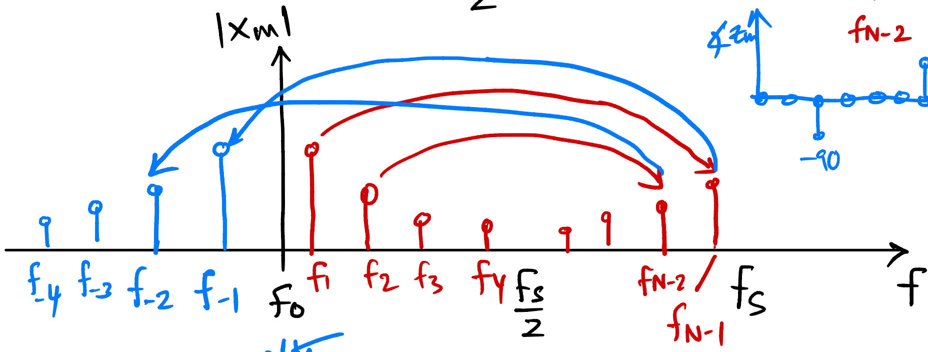
$$DFT(x[n] + y[n]) = DFT(x[n]) + DFT(y[n])$$
$$= X_m + Y_m$$

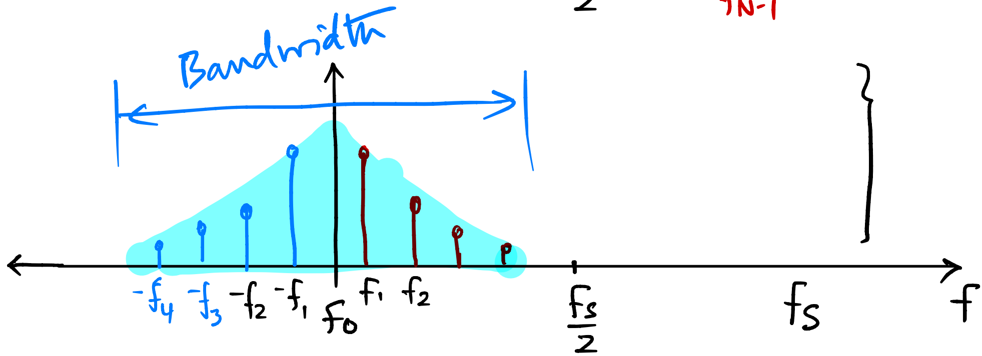
So what is DFT of $z[n] = \cos 2\pi f_1 n t_s + \sin 2\pi f_2 n t_s$?



② For real signals, DFT is symmetric

↳ Not only around $\frac{f_s}{2}$



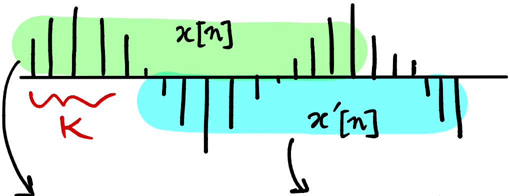


→ So take frequencies in $\left[\frac{f_s}{2}, f_s\right]$ and move to the negative freq. axis

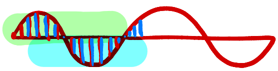
③ Its like the negative freq. sticks are going clockwise (i.e., (-)ve θ) to cancel the complex part.

③ DFT of a shifted signal is original DFT with a phase shift.

$$X'_m = X_m \cdot e^{j\frac{2\pi}{N}.m.K}$$



DFT $\equiv X_m$ DFT $\equiv X'_m$



Let's shift x by K samples

$$x'[n] = x[n+K]$$

$$X'_m = \text{DFT}\{x[n+K]\}$$

$$\therefore \quad X'_m = \sum_{n=0}^{N-1} x[n+k]\, e^{-j\frac{2\pi}{N}.m.n}$$

$$= \underbrace{\sum_{n=0}^{N-1} x[n+k]\, e^{-j\frac{2\pi}{n}m(n+k)}}_{\text{What is this?}} . e^{j\frac{2\pi}{N}.mk}$$

DFT of the shifted blue signal.

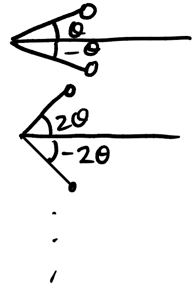
What is this? This is X_m which is DFT of green signal

$$\boxed{X'_m = X_m \underbrace{e^{j\frac{2\pi}{N}.m.K}}_{\text{means phase shift}}}$$

when $m = N-1$, $k=1$ $e^{j\theta.(N-1).1}$

This phase shift is proportional to K, and to frequency m as well.

$\frac{1}{2}\cos 2\pi F_1 n t_s$



Green Signal

$t_0 \quad = 1 = x_0$

$t_1 \quad = 2 \cdot \frac{1}{2}\cos\theta = x_1$

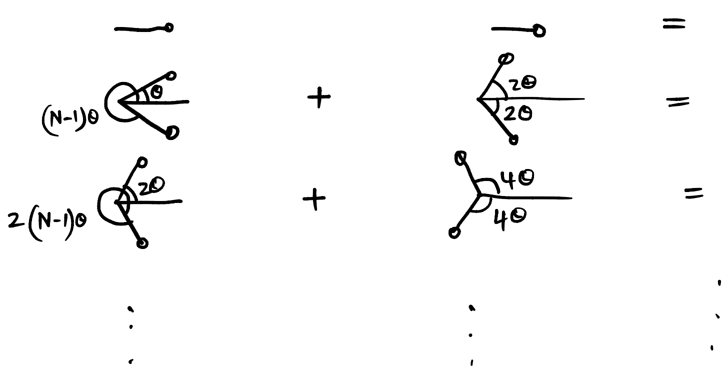
$t_2 \quad = 2 \cdot \frac{1}{2} \cdot \cos 2\theta = x_2$

$\vdots$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

$$x_n = \cos 2\pi f_1 n t_s + \cos 2\pi (2f_1) n t_s$$

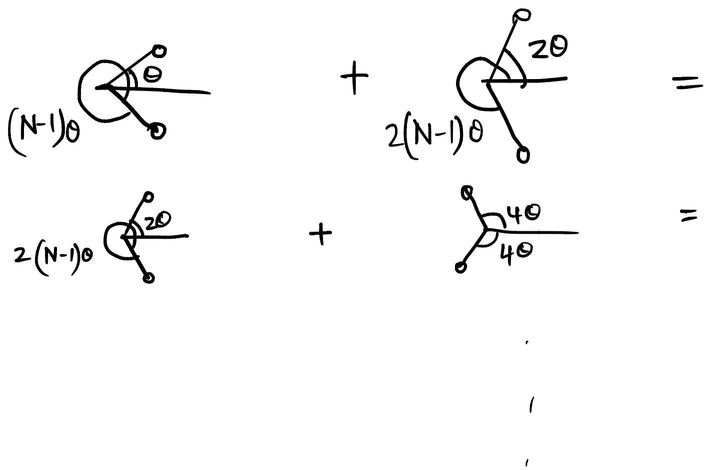
t_0, t_1, t_2, t_3



Now $z_1 e^{j\frac{2\pi}{N}\cdot(m=1)(k=1)}$ $\quad$ $z_2 e^{j\frac{2\pi}{N}\cdot(m=2)(k=1)}$

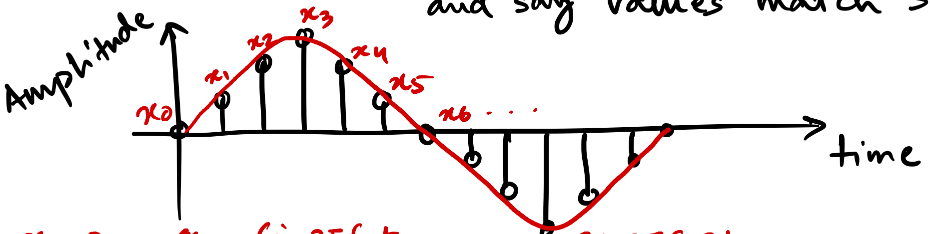
$z_{N-1}\cdot e^{j\frac{2\pi}{N}(m=N-1)(k=1)}$ $\quad$ $z_{N-2}\, e^{j\frac{2\pi}{N}(m=N-2)(k=1)}$

t_0, t_1



NYQUIST SAMPLING

- Say you are given the following discrete samples and asked to reconstruct the analog signal... and say values match sin(·)



$x_0 = 0$, $x_1 = \sin 2\pi f_1 t_s$, $x_2 = \sin 2\pi f_1 2t_s \ldots$

- We can say freq $f_1 = 10\ Hz$

  $\therefore$ Analog signal $= \sin(2\pi f_1 t)$ $\quad f_1 = 10Hz$

  $x[n] = \sin 2\pi f_1 n t_s$

- But is this the only signal that fits these values ?

  Observe : $\sin(2\pi f_1 n t_s + 2\pi m)$ also fit the given values so long as m is an integer.

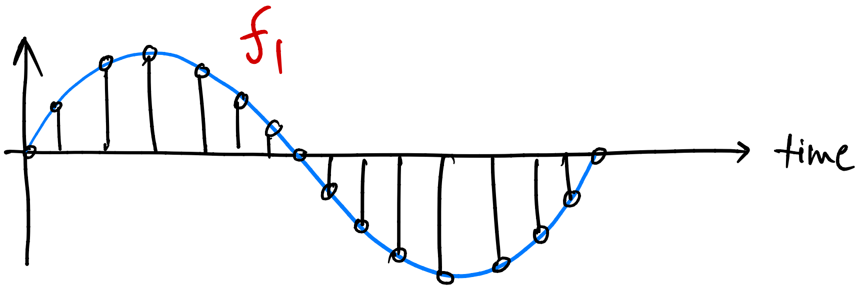
i.e., $\sin\left(2\pi\left(f_1 + \frac{m}{n t_s}\right) n t_s\right)$ fits values

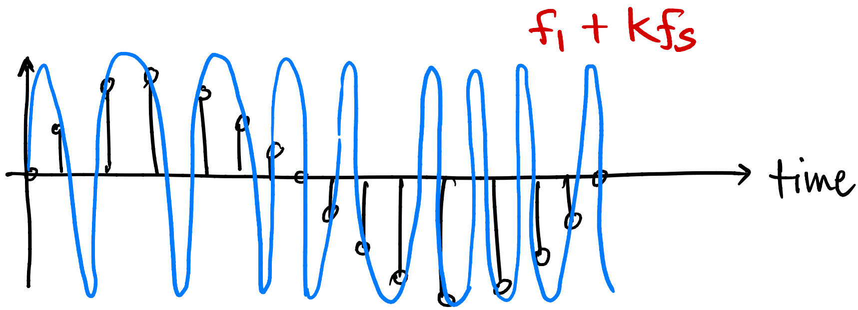
Choosing m as multiple of n $\left(\frac{m}{n} = k\right)$,

we have $\sin\left(2\pi\left(f_1 + k \cdot \frac{1}{t_s}\right) n t_s\right)$

$= \sin\left(2\pi \underline{(f_1 + k f_s)} n t_s\right)$ ... f_s is sampling freq.

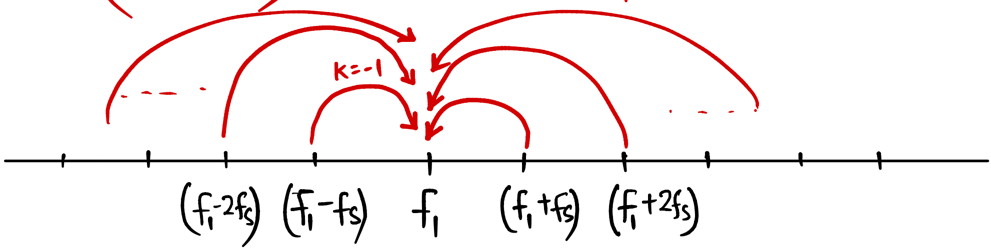
→ This means, many other freq, separated by kf_s from the core freq. f_1, fits the given set of points.



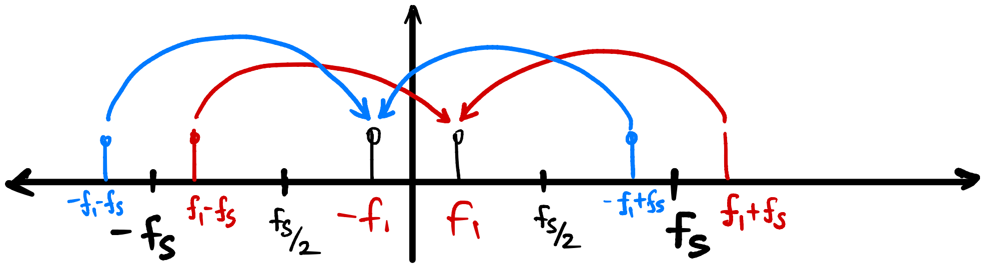


Called "ALIASED" frequencies

i.e., $(f_1 + kf_s)$ are aliases of f_1, for all values of k.



→ Observe that higher the sampling freq. f_s, greater is the seperation between the aliased freq.



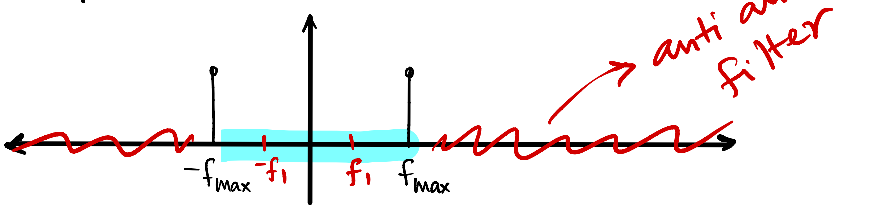
If I sample a signal at f_s, and therefore get to see freq. components of $\frac{f_s}{N}, \frac{2f_s}{N}, \frac{3f_s}{N} \ldots \frac{N-1}{N}f_s$,

actually, each of these freq. components could have aliased freq. from other freq contained in the signal.

$\Downarrow$

So what can we do?

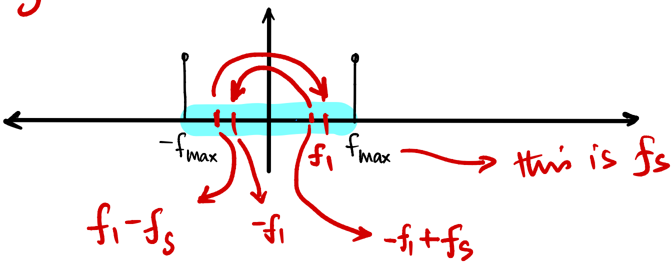
③ Now suppose you know



First easy step : Filter out freq > f_{max} and < $-f_{max}$.
Called anti-aliasing filter (it's an analog filter)

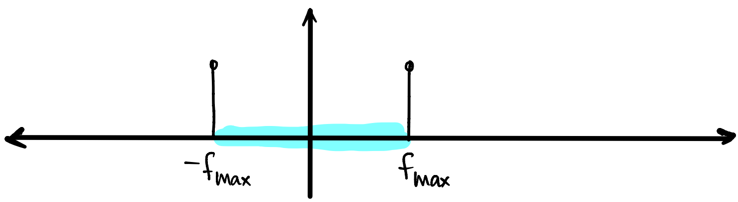
But is that enough ?

If sampling freq. $f_s \leq f_{max}$, freq. between $[-f_{max}, f_{max}]$ will alias into my bandwidth



Only way to prevent any freq. from $[-f_{max}, f_{max}]$ to alias to into my bandwidth is when

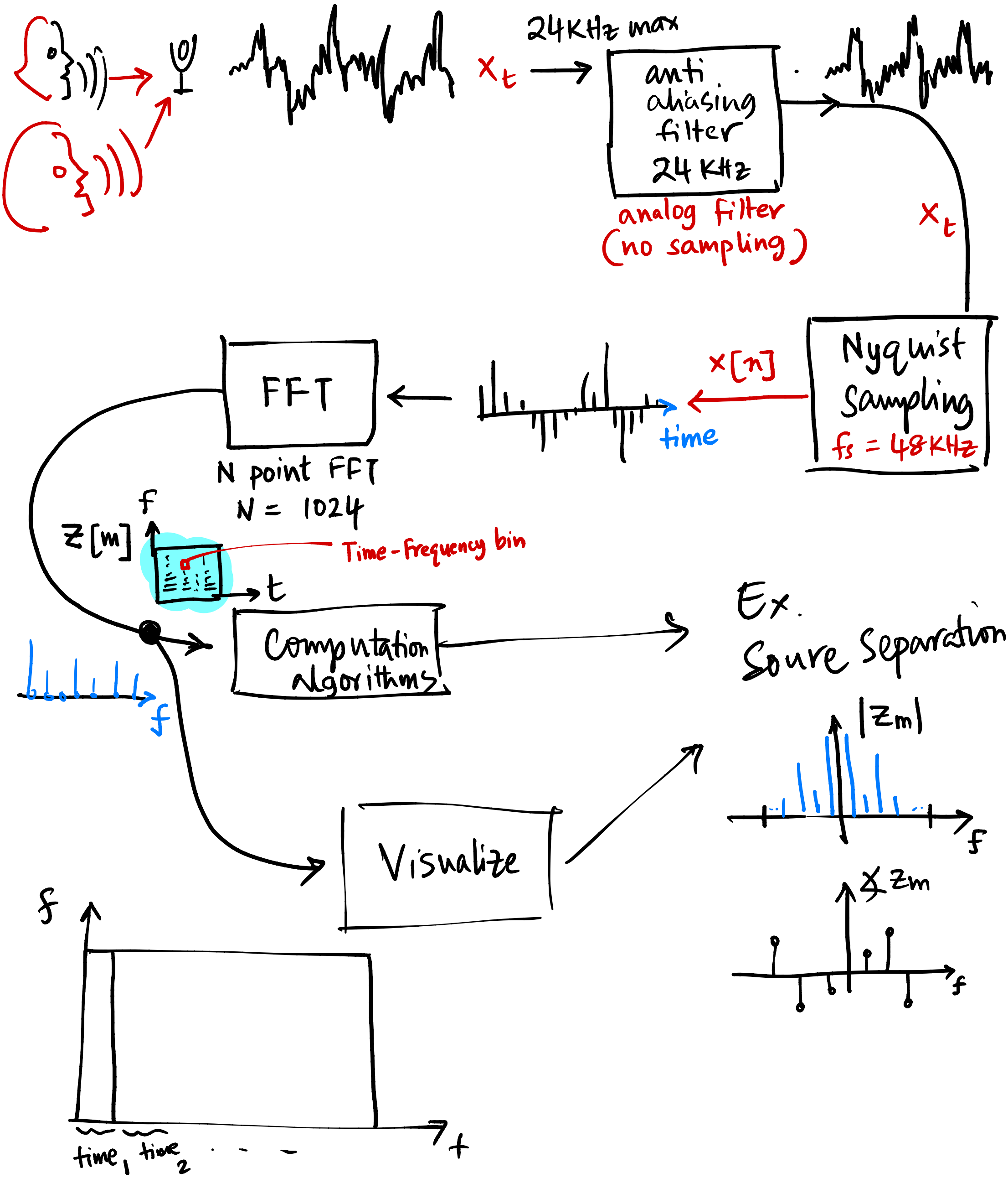
$f_{max} - f_s < -f_{max}$

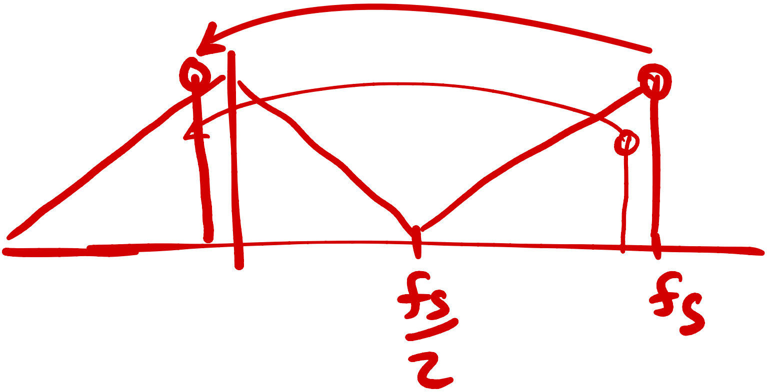


∴ $\boxed{f_s > 2 f_{max}}$ Nyquist's sampling rate.

sampling rate > twice the max freq. contained in the sig.

The signal processing pipeline:





aliasing $f_1 + k f_s$

$f_s \geq 2 f_{max}$