DSP : Discrete Fourier Transform (DFT)

#2

But observe that each, $\bar{f}_i$ vectors are N-Dimensional

And $f_0, f_1, f_2 \cdots f_{N-1}$ are all orthogonal,

thus must be also linearly indep.

$$\Downarrow$$

Vectors $\bar{f}_0, \bar{f}_1, \bar{f}_2 \cdots \bar{f}_{N-1}$ form a BASIS for N-dimensional space.

Now, express the original signal X in time & freq. basis

$$\underline{I\, X_t} = F\, X_f$$

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & & 1 & 0 \\ \vdots & & \ddots & \vdots \\ & & & 1 \end{bmatrix}}_{\text{Time Basis}} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}}_{\text{Sig. in time domain}} = \underbrace{\begin{bmatrix} | & | & & | \\ f_0 & f_1 & \cdots & f_{N-1} \\ | & | & & | \end{bmatrix}}_{\text{Fourier or frequency Basis}} \underbrace{\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{bmatrix}}_{\text{Fourier Transform of the sig. X}}$$

Thus, Fourier transform is the representation of a signal vector in a different (frequency) basis.

$$I X_t = F X_f \Rightarrow F \cdot X_f = X_t$$

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{N-1} \end{bmatrix} \leftarrow X_f = F^{-1} X_t = (F^*)^T X_t = \begin{bmatrix} — f_0^* — \\ — f_1^* — \\ — f_2^* — \\ \vdots \\ — f_{N-1}^* — \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

$$z_2 = e^{-j0(2\theta)} x_0 + e^{-j(2\theta)} \cdot x_1 + e^{-j2(2\theta)} \cdot x_2 + \dots e^{-j(N-1)(2\theta)} x_{N-1}$$

$$z_2 = \sum_{n=0}^{N-1} x_n e^{-jn(2\theta)} \quad \text{where } \theta = \frac{2\pi}{N}$$

$$\boxed{z_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-j \cdot \frac{2\pi}{N} m n}}$$

↳ Discrete Fourier Transform (DFT) ⇒ Fast ⇓ FFT

↳ What is 'm' in z_m? Speed or freq. of rotation

Inverse DFT ⇒ From freq back to time domain

$$I X = F . X_f \quad \text{given to you}$$

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ f_0 & f_1 & \cdots & f_{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{bmatrix}$$

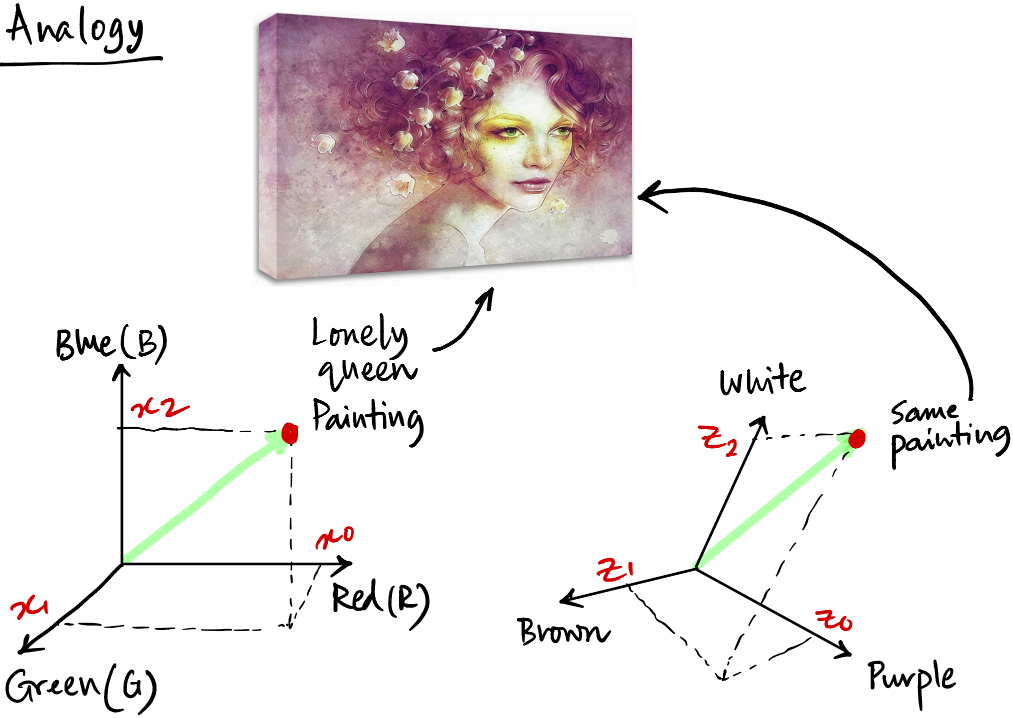
$$f_1 = \begin{bmatrix} e^{j0} \\ e^{j\theta} \\ e^{j2\theta} \\ \vdots \end{bmatrix}$$

$$x_1 = e^{j0} \cdot z_0 + e^{j\theta} \cdot z_1 + e^{j2\theta} \cdot z_2 \ \vdots \ + e^{j(N-1)\theta} z_{N-1}$$

$$\boxed{x_n = \sum_{m=0}^{N-1} z_m e^{j \frac{2\pi}{N} \cdot m \cdot n}}$$

↳ Inverse Discrete Fourier Transform (IDFT)

Analogy



$$\text{Lonely queen} = \begin{bmatrix} | & | & | \\ R & G & B \\ | & | & | \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ P & Br & W \\ | & | & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}$$

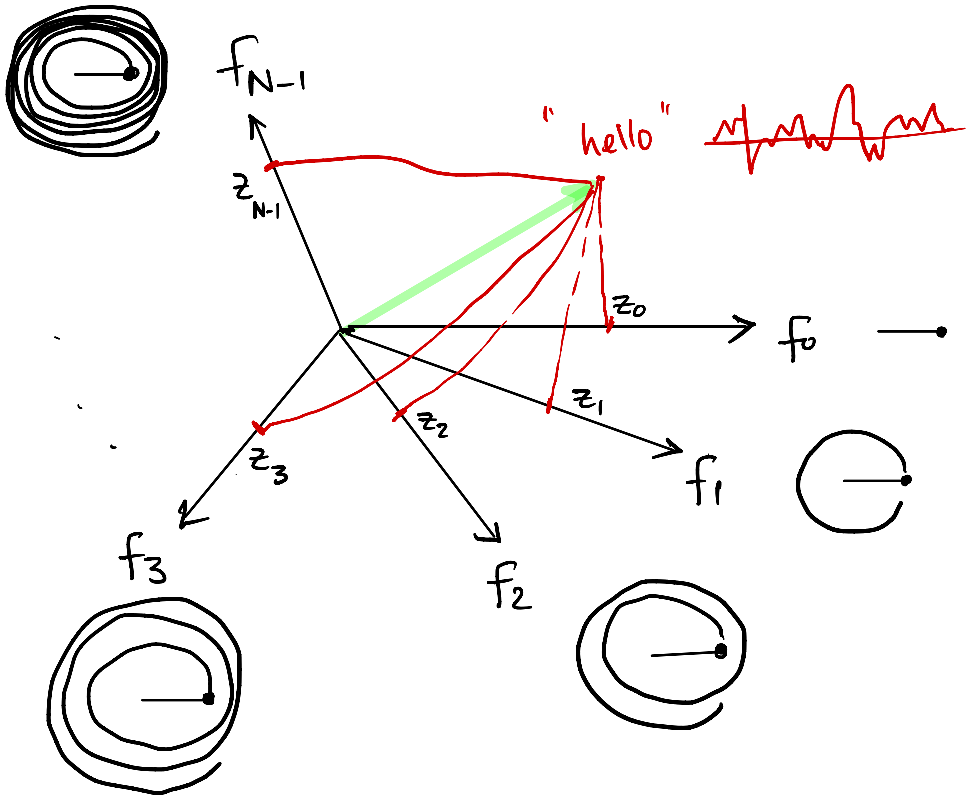
→ Now when basis is complex, what about the weights?

"hello"

$$\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}}_{\text{Real}} = \underbrace{\begin{bmatrix} e^{j0} & e^{j0} & \cdots & e^{j0} \\ e^{j0} & e^{j\theta} & & e^{j(N-1)\theta} \\ \vdots & \vdots & & \vdots \\ e^{j0} & e^{j(N-1)\theta} & & e^{j(N-1)^2\theta} \end{bmatrix}}_{\text{Fourier Basis} \equiv \text{Complex}} \underbrace{\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{bmatrix}}_{?\ \text{Complex vector}}$$

DFT is a complex Vector

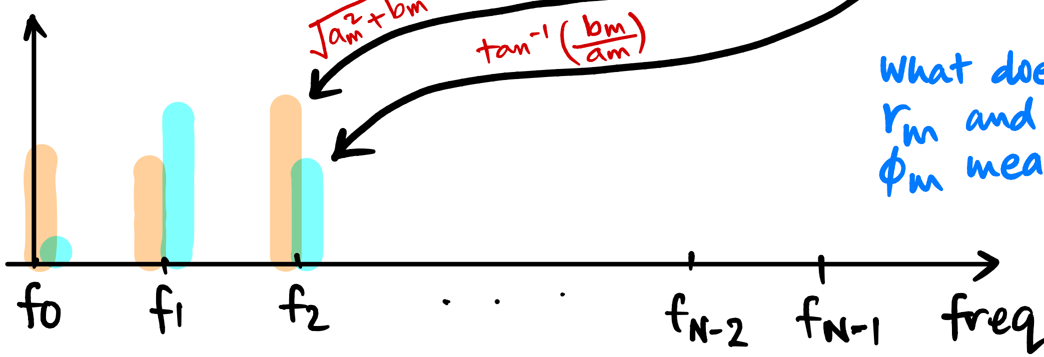
↳ What does that mean?



$$z_m = a_m + j b_m = r_m e^{j\phi_m}$$

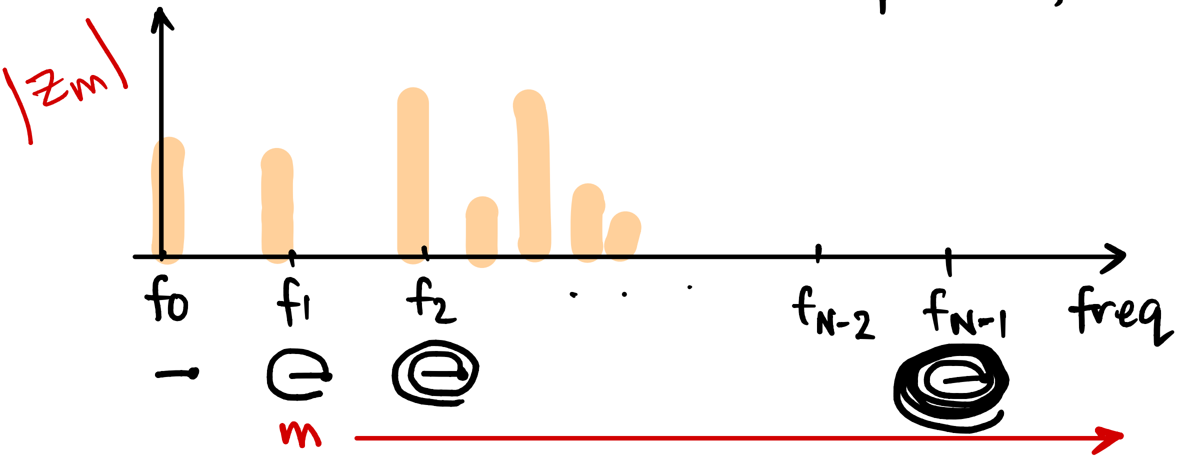
$\sqrt{a_m^2+b_m^2}$

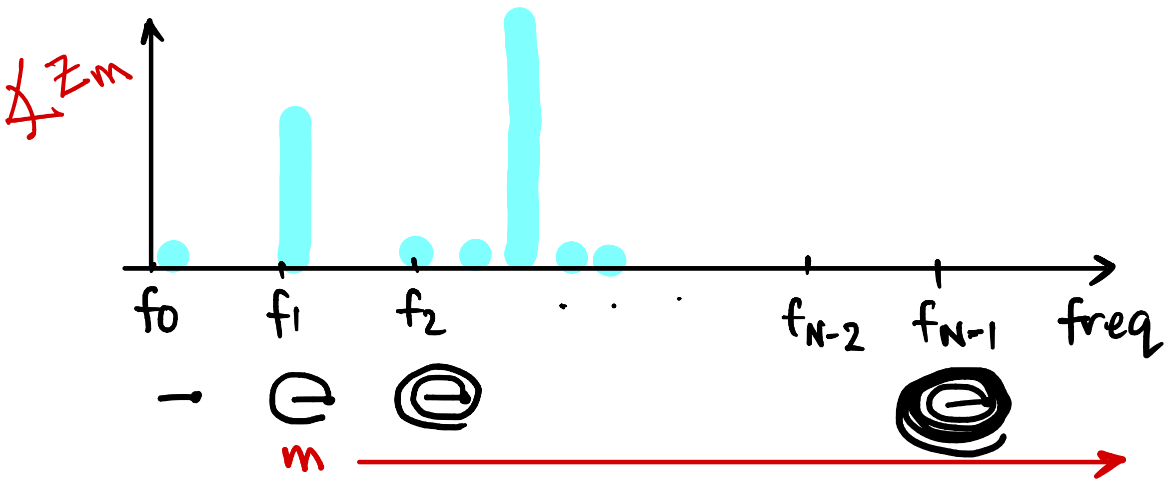
$\tan^{-1}\left(\frac{b_m}{a_m}\right)$



What does r_m and ϕ_m mean?

Magnitude and Phase plots of DFT

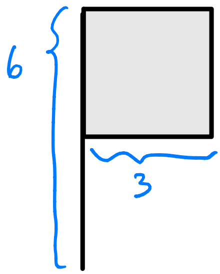




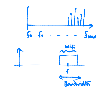
→ signals will have non-zero magnitudes for some frequencies.

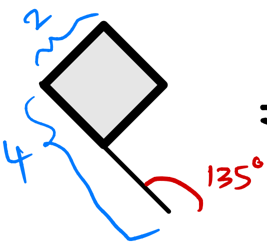
Bandwidth = max non-zero freq. contained in a given signal.

Analogy



$= \; | \times 6 + \square \times 3$



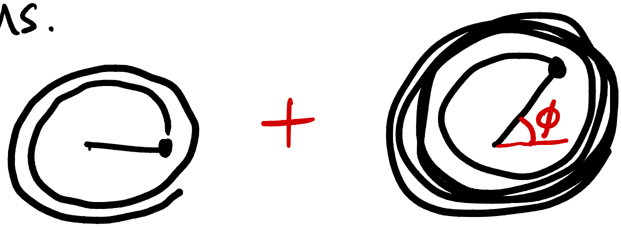


$= \; | \times 4e^{j135} + \square \times 2e^{j45}$

Rowit transform $= \begin{bmatrix} 4e^{j135} \\ 2e^{j45} \end{bmatrix}$

→ Similarly, to synthesize any signal with spinning sticks, you may need to start some of the sticks by rotating them with phase ϕ_M, in addition to increasing/decreasing their lengths.

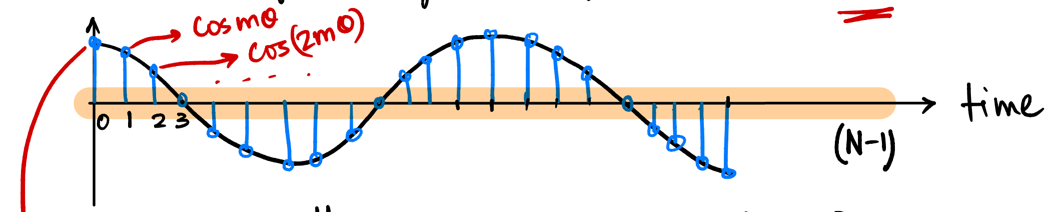
signal we want =



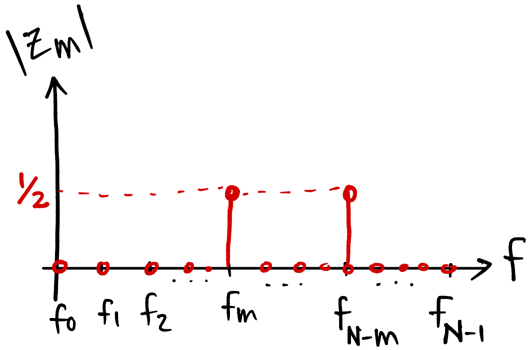
Q: Does a time domain signal have phase?

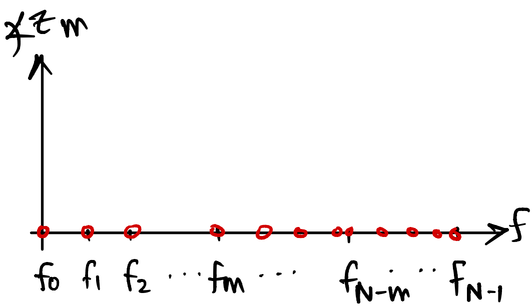
Example 1 : $x(t) = \cos 2\pi f_m t \rightarrow nt_s$

Discrete sampled signal $x[n] = \cos 2\pi f_m n t_s$



⇓ synthesize with frequency basis.





→ But $\cos 2\pi f_m n t_s$ only has f_m freq.
Why does DFT have 2 freq. components?

Answer:

: Note that $\cos 2\pi f_m n t_s$ is not a rotation.
It's only the x-axis shadow of the rotation.
So, to make $\cos 2\pi f_m n t_s$, we need to cancel out the y-axis shadows of the rotating stick.

$\cos 2\pi f_m n t_s$

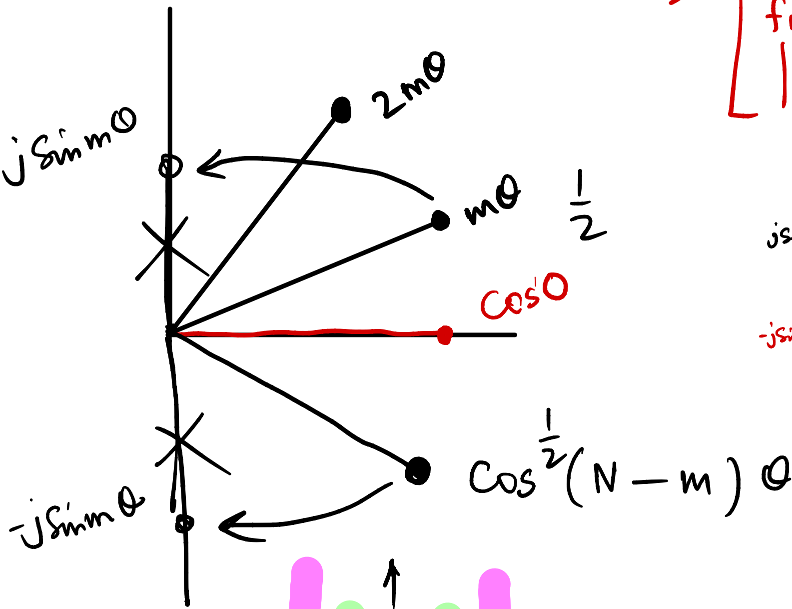
$$\begin{bmatrix} \cos 0 \\ \cos(m\theta) \\ \cos 2(m\theta) \\ \cos 3(m\theta) \\ \vdots \\ \cos(N-1)(m\theta) \end{bmatrix} = \begin{bmatrix} e^{j0} \\ e^{jm\theta} \\ e^{j2m\theta} \\ e^{j3m\theta} \\ \vdots \\ e^{j(N-1)\theta} \end{bmatrix} \times z_m$$

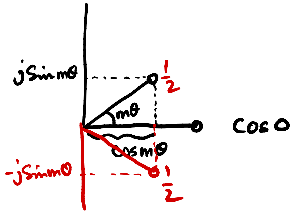
$\cos 0 + \cancel{j\sin 0}$

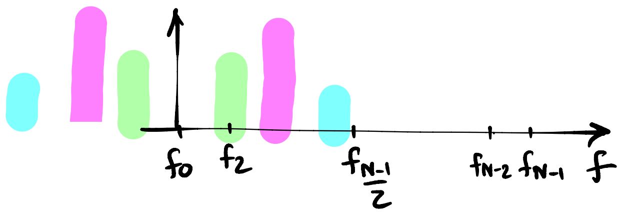
$\cos m\theta + j\sin m\theta$

$\cos 2m\theta + j\sin 2m\theta$

$$\begin{bmatrix} | \\ f_m \\ | \end{bmatrix}$$







⊛ Thus, real signals always symmetric around center of the whole frequency band $[f_0, f_{N-1}]$

⊛ For $\cos 2\pi f_m n t_s$, all phases are 0

$$\begin{bmatrix} \cos 0 \\ \cos m\theta \\ \cos 2m\theta \\ \vdots \\ \cos (N-1)m\theta \end{bmatrix}$$