Eigenvalues and Eigenvectors.

$A\bar{x} = b$ or $\bar{x} \rightarrow \boxed{A(\cdot)} \rightarrow \bar{b}$

A: function or operation; $\bar{x}$: input; b: output

$A_{m \times n}$

$x \in n \times 1$

$b \in m \times 1$

$A\bar{x} = \lambda \bar{x}$

A: square matrix; λ: scalar multiple

input vector only changes in mag. but not in direction ...

$x \equiv$ eigenvectors of A, $x \neq 0$

$\lambda \equiv$ eigenvalue of A.

How to obtain eigenvectors and eigenvalues?

$Ax = \lambda x \Rightarrow A\bar{x} - \lambda \bar{x} = 0 \Rightarrow (A - \lambda I)x = 0, \; x \neq 0$

$\det(A - \lambda I) = 0$

Null space of $(A - \lambda I)$ exists.

e.g., $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\therefore (A - \lambda I) = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

Determinant$(A - \lambda I) = (3-\lambda)^2 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow (\lambda - 4)(\lambda - 2) = 0$

$\therefore \lambda_1 = 4, \quad \lambda_2 = 2 \Rightarrow$ 2 eigenvalues of A.

Now, $\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(A - \lambda_1 I)$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad (A - \lambda_2 I)$ $\quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Eigen Decomposition and Diagonalization

$$\left.\begin{aligned} A x_1 &= \lambda_1 x_1 \\ A x_2 &= \lambda_2 x_2 \\ &\vdots \\ A x_n &= \lambda_n x_n \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} \\ & A & \\ \\ \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}}_{\Lambda}$$

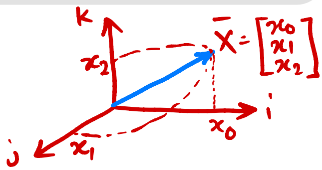
$$\therefore \quad AS = S\Lambda$$

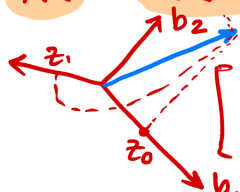
$\Rightarrow \quad A = S\Lambda S^{-1}$ → called **Eigen decomposition**

$\Rightarrow \quad S^{-1}AS = \Lambda$ → called **Diagonalization**

Change of basis

$$Ax = S\Lambda S^{-1}x$$





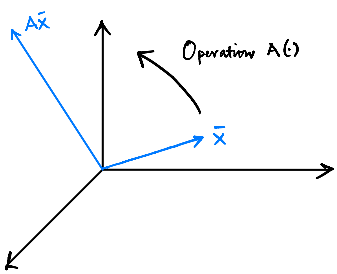
$$\begin{bmatrix} | & | & | \\ i & j & k \\ | & | & | \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ b_0 & b_1 & b_2 \\ | & | & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}$$

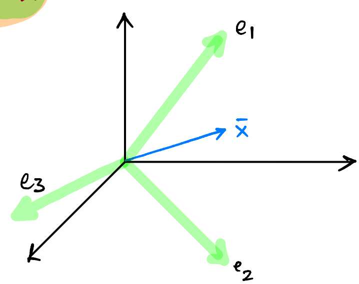
$$I \cdot \bar{x} = B \cdot \bar{z}$$

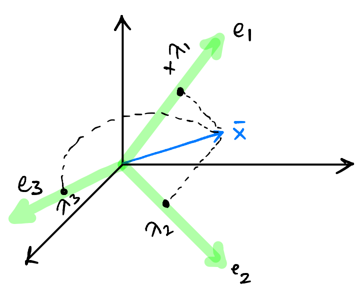
$$\bar{z} = B^{-1}\bar{x}$$

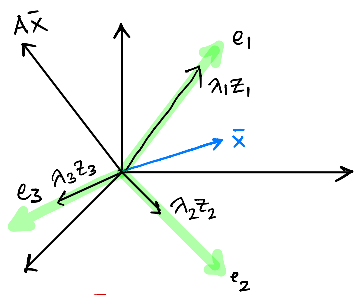
$$S\Lambda S^{-1}x$$

Intuition for $A = S\Lambda S^{-1}$

① 

② 

③ 

④ 

So what is A^2 ? A^3 ? A^n ?

$$A^{100}x \quad \ldots A(A(Ax))$$

$$Ax = S\Lambda S^{-1}x$$

$$A(Ax) = S\Lambda S^{-1}(S\Lambda S^{-1})x$$

$$= S\Lambda \underbrace{S^{-1}S}_{I} \Lambda S^{-1}x$$

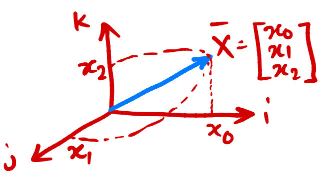
$$= S\Lambda\Lambda S^{-1}x$$

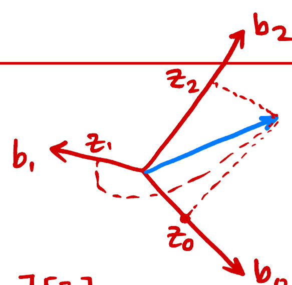
$$= S\Lambda^{2}S^{-1}x.$$

$$A^{n}x = S\Lambda^{n}S^{-1}x$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 10 \\ 80 \\ 120 \\ 95 \end{bmatrix}}_{x} = \begin{bmatrix} 105 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\lim_{n\to\infty} A^{n}x$$





$$\begin{bmatrix} | & | & | \\ i & j & k \\ | & | & | \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ b_0 & b_1 & b_2 \\ | & | & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}$$

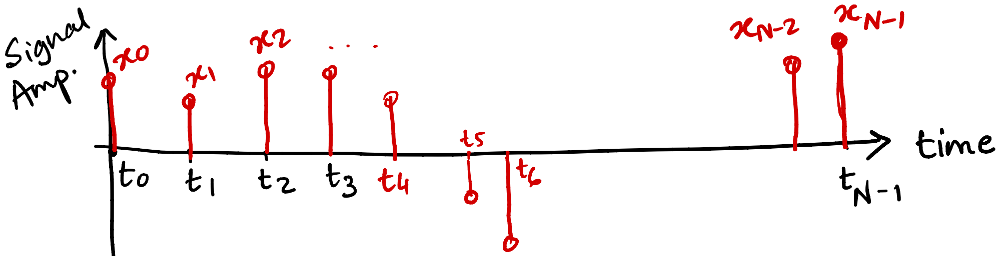
$$I . \bar{X} = B . \bar{Z}$$

$$\bar{Z} = B^{-1}\bar{X}$$

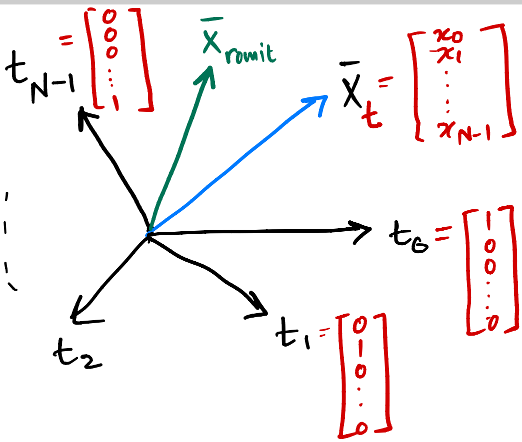
DSP : Discrete Fourier Transform (DFT)

#1

"Hello"

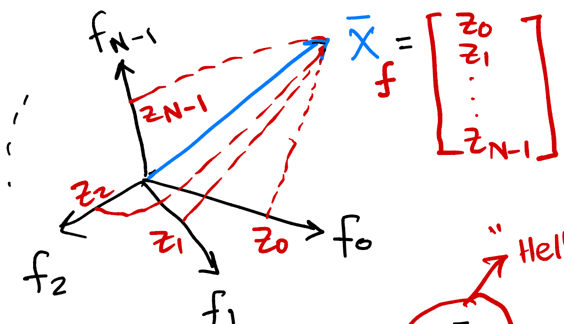


$$\bar{X} = [x_0 \; x_1 \; x_2 \; \cdots \; x_{N-1}]^T$$



$$\bar{X} = x_0 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}}_{t_0} + x_1 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}}_{t_1} \cdots + x_{N-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{t_{N-1}}$$

$$\bar{X} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ 0 & \vdots & & \vdots \\ \vdots & 1 & & 1 \end{bmatrix}}_{\text{Basis Vectors (time basis)}} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}}_{\text{Signal in the time basis.}}$$

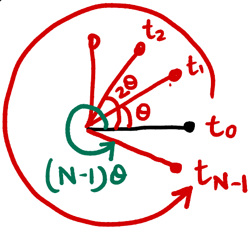


$$\bar{X}_t = \underbrace{\begin{bmatrix} | & | & & | \\ f_0 & f_1 & \cdots & f_{n-1} \\ | & | & & | \end{bmatrix}}_{\text{Some other basis (F)}} \underbrace{\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{bmatrix}}_{\text{same hello expressed in this F basis}}$$

"Hello"

Thus, $I \bar{X}_t = F . X_f$

MODEL ROTATION



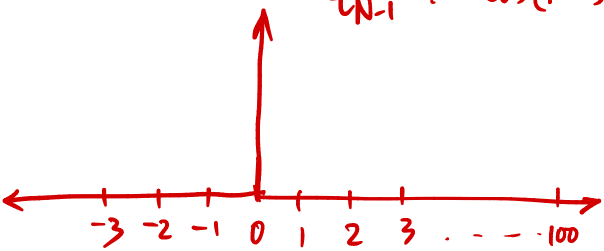
$$t_0 : 1 \equiv e^{j0}$$

$$t_1 : \cos\theta + j\cancel{1}\sin\theta \equiv e^{j\theta}$$

$$t_2 : \cos 2\theta + j\cancel{1}\sin 2\theta \equiv e^{j2\theta}$$

$$\vdots$$

$$t_{N-1} : \cos(N-1)\theta + j\cancel{1}\sin(N-1)\theta \equiv e^{j(N-1)\theta}$$



$$j \cdot 1 = j$$

$$j(j \cdot 1) = -1$$

$$j^2 = -1$$

complex number ← $\boxed{j = \sqrt{-1}}$

$\therefore$ Rotation of a stick of length 1 $\equiv \left[e^{j0} \ e^{j\theta} \ e^{j2\theta} \ \cdots \ e^{j(N-1)\theta} \right]^T$

Now, how can I model different speeds/frequency of rotation?

0 cycles/ N time steps | 1 cycle/ N time steps | 2 cycles/ N time steps | N-1 cycles/ N time steps

$$\begin{bmatrix} e^{j0} \\ e^{j0} \\ e^{j0} \\ \vdots \\ e^{j0} \end{bmatrix} \quad \begin{bmatrix} e^{j0} \\ e^{j\theta} \\ e^{j2\theta} \\ \vdots \\ e^{j(N-1)\theta} \end{bmatrix} \quad \begin{bmatrix} e^{j0} \\ e^{j2\theta} \\ e^{j2(2\theta)} \\ \vdots \\ e^{j(N-1)(2\theta)} \end{bmatrix} \quad \cdots \quad \begin{bmatrix} e^{j0} \\ e^{j(N-1)\theta} \\ e^{j2(N-1)\theta} \\ \vdots \\ e^{j(N-1)^2\theta} \end{bmatrix}$$

Freq