Linear Algebra

#3

Question: How many sol^{n}s possible to $A\bar{x} = \bar{b}$, and what dim. is $N(A)$?

(a)

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \text{Square} \\ \text{matrix} \end{bmatrix} \end{array}$$

Rank $= m = n$

Full rank

x has 1 solution

$N(A)$ is ϕ

$N(A^T)$ is ϕ

(b)

$$\begin{array}{cc} & n=3 \\ m=5 & \begin{bmatrix} \text{Thin} \\ \text{matrix} \end{bmatrix} \end{array}$$

Rank $= n < m$

Full col. rank

x has $\underline{0 \text{ or } 1}$ sol^{n}s.

if $b \in C(A)$

but if $b \notin C(A)$

$N(A)$ is ϕ

$N(A^T)$ is $\mathbb{R}^2$

(c)

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \text{fat} \\ \text{matrix} \end{bmatrix} \end{array}$$

Rank $= m < n$

Full row rank

x has $\underline{\infty}$ sol^n.

$N(A)$ is $\mathbb{R}^{n-m}$

$N(A^T)$ is ϕ

To see this, turn this A^T matrix to a thin matrix

(d)

$$\begin{array}{cc} & n=4 \\ m=5 & \begin{bmatrix} \text{Matrix} \end{bmatrix} \end{array}$$

$r=3$

Rank $=r < m$, Rank $< n$

Rank deficient matrix

x has $\underline{0\ (b \notin C(A)),\ \infty\ (b \in C(A))}$ sol^n.

$N(A)$ is $n-r$

$N(A^T)$ is $m-r$

- **Basis** : Linearly Independent vectors that span a space
- **Dimensions** = # of vectors in basis = | Basis |
  - A space can have multiple bases but all such bases have equal dimension

Q: $\dim(C(A)) = ?$ Rank of A = $\dim(R(A))$

- **Orthogonal vectors** : $\bar{x} \cdot \bar{y} = x^T y = 0$ then $\bar{x}$ and $\bar{y}$ are orthogonal

Q: wall and floor orthogonal ?
- NOT orthogonal bcos of vectors living at their intersection

- **Norm** : $m\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \longrightarrow$ scalar $\Rightarrow \mathbb{R}^m \rightarrow \mathbb{R}$

$w = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

$\|w\|_2 = \sqrt{1^2+3^2+5^2}$

$$\|\bar{v}\|_p = \left(\sum_{i=1}^{m} |v_i|^p\right)^{1/p}$$ called p-norm

- **Length** = L_2 norm = $\sqrt{v_1^2+v_2^2+\ldots v_m^2} = \sqrt{v^T v}$
- **L_0 norm** = # of elements in the vector
- **uv^T** = $\begin{bmatrix} | \\ u \\ | \end{bmatrix}\begin{bmatrix} — v^T — \end{bmatrix} = \begin{bmatrix} & & \end{bmatrix}$ = Rank 1 matrix

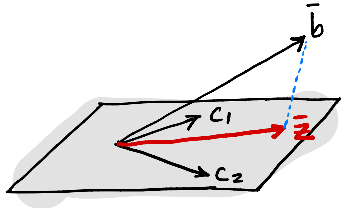
$\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$

$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- **Symmetric matrix** ≡ $A = A^T$
- **$(AB)^{-1}$** = $B^{-1}A^{-1}$ and $(AB)^T = B^T A^T$
- **A^TA** is a square and symmetric matrix

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} =$

Q: But say you have to solve $A\bar{x} = \bar{b}$ even though $\bar{b} \notin C(A)$. What's the best you can do?

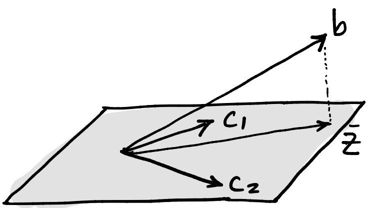


Ans: Produce a vector as $\bar{z} = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

s.t. $\bar{z}$ is closest possible to $\bar{b}$.

Now declare $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the approx solⁿ to $A\bar{x} = \bar{b}$

What is "closest possible"?



Define closest possible as minimum length vector $(\bar{b} - \bar{z})$ ⇒ or least square error

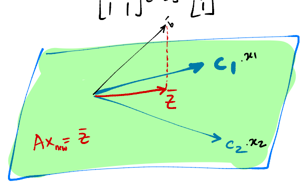
i.e., $\bar{z}$ is projection of $\bar{b}$ on $C(A)$

Given this, find solution to $\bar{x}$ which is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$X = \bar{b}$ $\begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} | \\ b \\ | \end{bmatrix}$



x, x_{new} both vectors.

$$x_{new} = \frac{A^T b}{A^T A}$$

$$Ax = b$$

$$\begin{bmatrix} \vdots \end{bmatrix} x = \begin{bmatrix} b \end{bmatrix}$$

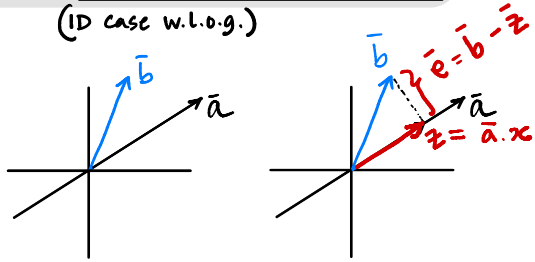
Least Squares Solution

(1D case w.l.o.g.)

$Ax = b$

$\downarrow$

$\bar{a} = \begin{bmatrix} \vdots \end{bmatrix}$



$$\bar{z} = x \cdot \bar{a}$$

$$\bar{z} + \bar{e} = \bar{b}$$

$$\bar{e} = \bar{b} - \bar{z} = \bar{b} - \bar{a}.x$$

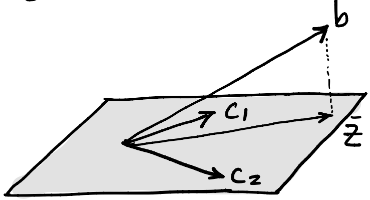
Now, since $\bar{a} \perp \bar{e}$, we have $a^T e = 0$

or $\quad a^T(\bar{b} - \bar{a}x) = 0 \quad$ or $\quad a^T b - a^T a x = 0$

$\therefore \quad x = \dfrac{a^T b}{a^T a} \quad \leftarrow$ A is a vector $\bar{a}$

Now, take this to higher dimensions

say 2D.



$$\bar{x} = \frac{A^T b}{A^T A} = \underbrace{(A^T A)^{-1} A^T b}$$

Projected vector:

$$\bar{z} = A\bar{x} = A.$$

$$= \frac{AA^T b}{(A^T A)}$$

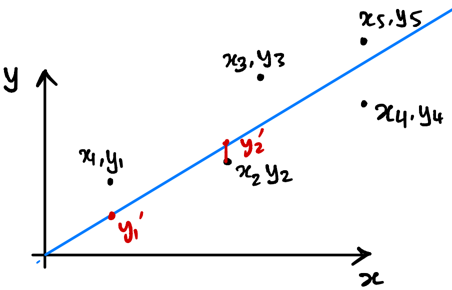
Projection of $\bar{b}$ onto $C(A)$

$\therefore$ Projection matrix $(P) = \dfrac{AA^T}{(A^T A)}$

Matrix P projects any vector to the column space of A.

green $b \rightarrow \boxed{\text{orange}} \rightarrow \bar{z}$

Example : Regression ⇒ Which line $\boxed{y = mx + c}$ satifies all points.



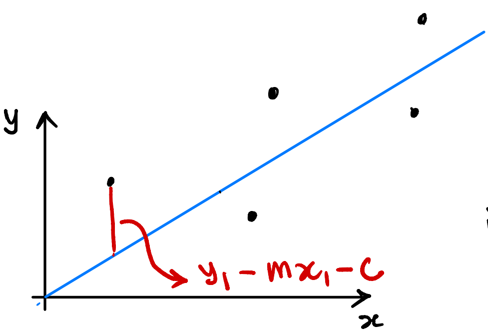
$$
\begin{aligned}
y_1 &= m x_1 + c \\
y_2 &= m x_2 + c \\
&\vdots \\
y_n &= m x_n + c
\end{aligned}
$$

$$
\Rightarrow \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}
$$

$\bar{z}$

Since no solution exists ⇒ $b \notin C(A)$

so minimize error $e_i = \bar{b} - A\bar{x} = y_i - (mx_i + c)$



For this minimization, we know solution $\bar{x}$

i.e., $\bar{x} = \dfrac{A^T b}{(A^T A)}$

$$
\boxed{\begin{bmatrix} m \\ c \end{bmatrix} = \frac{A^T b}{A^T A} = \left(\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right)}
$$

↪ Least Square Solution

Eigenvalues and Eigenvectors.

$A\bar{x} = b$ or $\bar{x} \rightarrow \boxed{A(\cdot)} \rightarrow \bar{b}$

A: function or operation; $\bar{x}$: input; b: output

$A_{m \times n}$

$x \in n \times 1$

$b \in m \times 1$

$A\bar{x} = \lambda \bar{x}$

A: square matrix; λ: scalar multiple

input vector only changes in mag. but not in direction ...

$x \equiv$ eigenvectors of A, $x \neq 0$

$\lambda \equiv$ eigenvalue of A.

How to obtain eigenvectors and eigenvalues?

$$Ax = \lambda x \Rightarrow A\bar{x} - \lambda\bar{x} = 0 \Rightarrow (A - \lambda I)x = 0, \; x \neq 0$$

$$\det(A - \lambda I) = 0$$

Null space of $(A - \lambda I)$ exists.

e.g.,

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \therefore (A - \lambda I) = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$\operatorname{Det}(A - \lambda I) = (3-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow (\lambda - 4)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 4, \quad \lambda_2 = 2$$

Now,

$$\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigen Decomposition and Diagonalization

$$\left.\begin{aligned} A x_1 &= \lambda_1 x_1 \\ A x_2 &= \lambda_2 x_2 \\ &\vdots \\ A x_n &= \lambda_n x_n \end{aligned}\right\}$$

$$\therefore \quad AS = S\Lambda$$

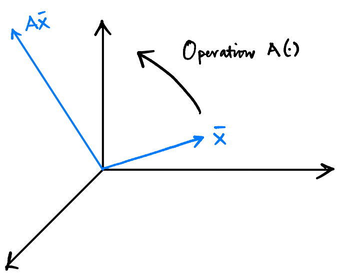
$\Rightarrow \quad A =$ → called

$\Rightarrow \quad AS =$ → called

Change of basis

Intuition for $A = S\Lambda S^{-1}$

1



2