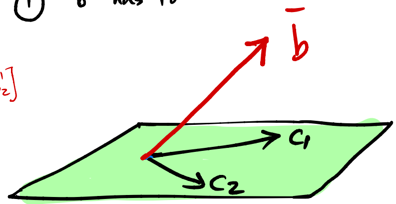
Linear Algebra

#2

$\rightarrow$ $A\bar{x} = \bar{b}$: when is this solvable? 2 conditions need to hold.

① $\bar{b}$ has to

$\begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$



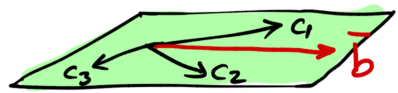
$\Downarrow$

- No way to
- Any $A\bar{x}$

② A has to be invertible

$\Downarrow$

There shouldn't be multiple ways of getting to $\bar{b}$ from columns of A.



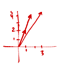
$\Downarrow$

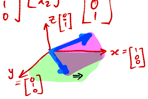
- Matrix is singular $\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}$
- Matrix is non-invertible
- Determinant (A) = 0

$\begin{bmatrix} 3 & 8 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 1 & \cancel{1} \ 2 \\ \cancel{1} \ 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$





$\rightarrow$ Rank(A) = No. of linearly independent columns (or rows) of that matrix.

$\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} \Rightarrow$ linearly dep. if

$$w_1 c_1 + w_2 c_2 + w_3 c_3 = 0$$
$$w_1 c_1 + w_2 c_2 = -w_3 c_3$$
$$\frac{w_1}{-w_3} c_1 + \frac{w_2}{-w_3} c_2 = c_3$$

means $\bar{c}_3$ can be expressed as

$$\begin{bmatrix} 9 \ \cancel{3} & \cancel{6} 0 & 9 \\ -15 \ \cancel{5} & + \ \cancel{4} 0 \ + & 15 \\ -6 \ \cancel{2} & \cancel{2} 0 & 6 \\ -3 \ \cancel{1} & \cancel{2} 0 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$w_1 = -3$
$w_2 = 0$
$w_3 = 1$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \neq 0$$

➔ **Null space** : $\{\bar{x} : A\bar{x} = 0\}$ ≡ space of all non-zero vectors that can take a matrix A to 0
$N(A)$

➔ **shape of null space** :

$n = 2$

$$A \rightarrow \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

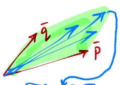
$m = 3$

m

$$A^T \rightarrow \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

n

$\downarrow C(A)$ $\quad \downarrow N(A) \quad \bar{q} \;\; \bar{p} \quad \downarrow C(A^T) \quad \downarrow N(A^T)$



$A(w_1\bar{q} + w_2\bar{p})$
$= A.w_1\bar{q} + Aw_2.\bar{p}$
$= w_1.\underbrace{A\bar{q}}_{0} + w_2\underbrace{A.\bar{p}}_{0}$
$=$

Observe that :

$\overline{c_{11} \;\; c_{21} \;\; c_{31}} \cdot \overline{y_1 \;\; y_2 \;\; y_3}$

$\overline{c_{12} \;\; c_{22} \;\; c_{32}} \cdot \overline{y_1 \;\; y_2 \;\; y_3}$

$$\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \,?$$

∴ $N(A^T) \perp$ and $N(A) \perp$

$$\begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \,?$$

➔ **Orthogonal vectors** :

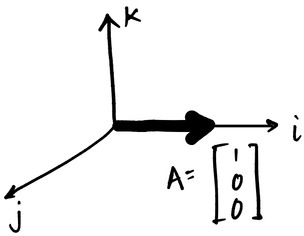
Two vectors $\bar{u}$ and $\bar{v}$ are orthogonal (perp.) if $\bar{u} \cdot \bar{v} = 0$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$u^T v = 0$

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 = 0$$

How large is $N(A^T)$?



T/F ?
Null space of this matrix A contains vector $\bar{j}$ and $\bar{k}$

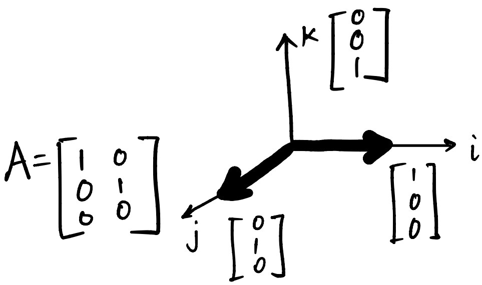
False

T/F : Null space of A^T contains $\bar{j}$ and $\bar{k}$

True.

$$A^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$



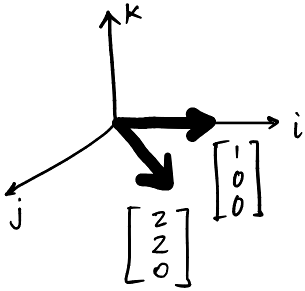
T/F ?
Null space of $A = \begin{bmatrix} | & | \\ i & j \\ | & | \end{bmatrix}$ contains $\bar{k}$

False

- Null space of A^T contains $\bar{k}$

True

$A=\begin{bmatrix}1&2\\0&2\\0&0\end{bmatrix}$



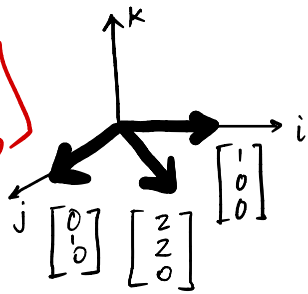
T/F ? $N(A)$ does not exist

True

T/F ? $N(A^T)$ contains $\bar{k}$

True

$A=\begin{bmatrix}1&2&0\\0&2&1\\0&0&0\end{bmatrix}$



T/F ? $N(A)$ does not exist

False

T/F ? $N(A^T) = \phi$

False because $\bar{k} \perp C(A)$

also. $A^T k = 0$

$A_{6\times 2} = \begin{bmatrix}1&3\\15&8\\7&1\\3&1\\22&9\\8&13\end{bmatrix}$

$|C(A)|=2$

$C(A) = \mathbb{R}^2$

T/F ? $N(A) = \phi$

True ?

T/F ? $N(A^T) = \phi$

False.

Now how many dimensions in $N(A^T)$?

$A^T = \begin{bmatrix}1&15&7&3&22&8\\3&8&1&1&9&13\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_6\end{bmatrix} = 0$

Many vectors exist

Think of this as 2D space covered by 6 vectors

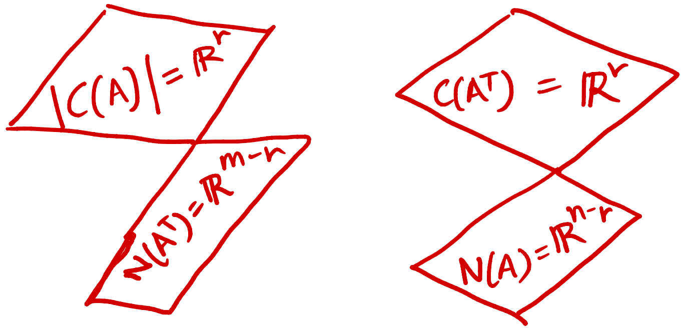
$C(A) = \mathbb{R}^2$ $N(A^T) = \mathbb{R}^4$ $C(A^T) = \mathbb{R}^2$ $N(A) = \phi$

Generalize

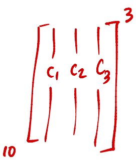
$(A_{m \times n})$

rank $= r$

$r \leq \min\{m, n\}$



Intuition :



Think of $N(A^T)$ as : the gap between the potential space the columns could cover and the space it covered.

Think of $N(A)$ as the space of redundancy or inefficiency of the columns.

- Add m-dimensional cols. one by one to fill out as much space