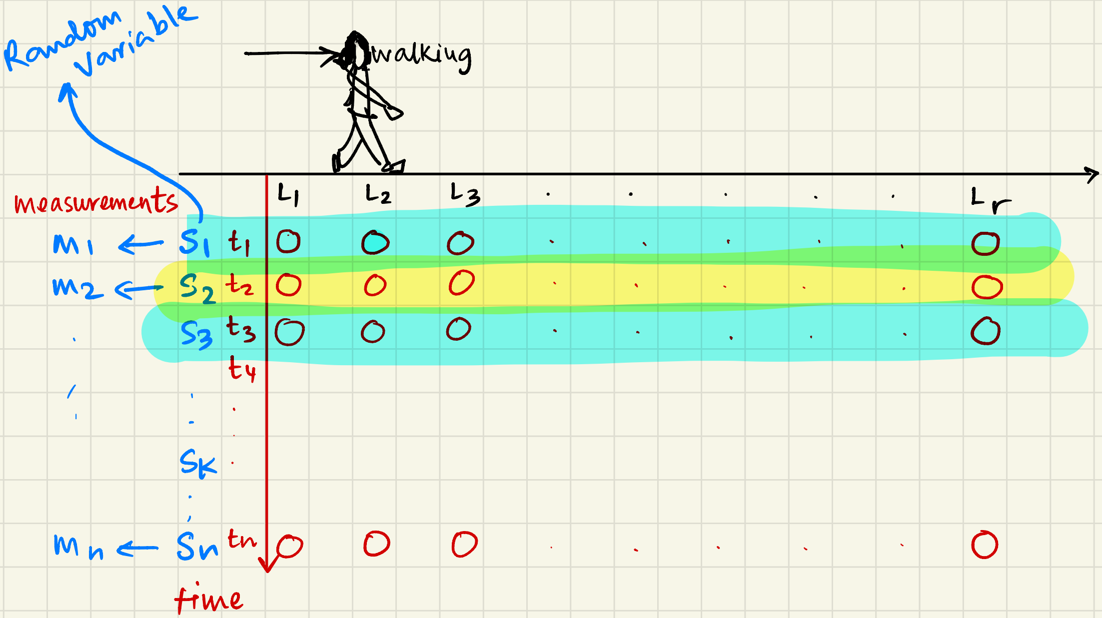
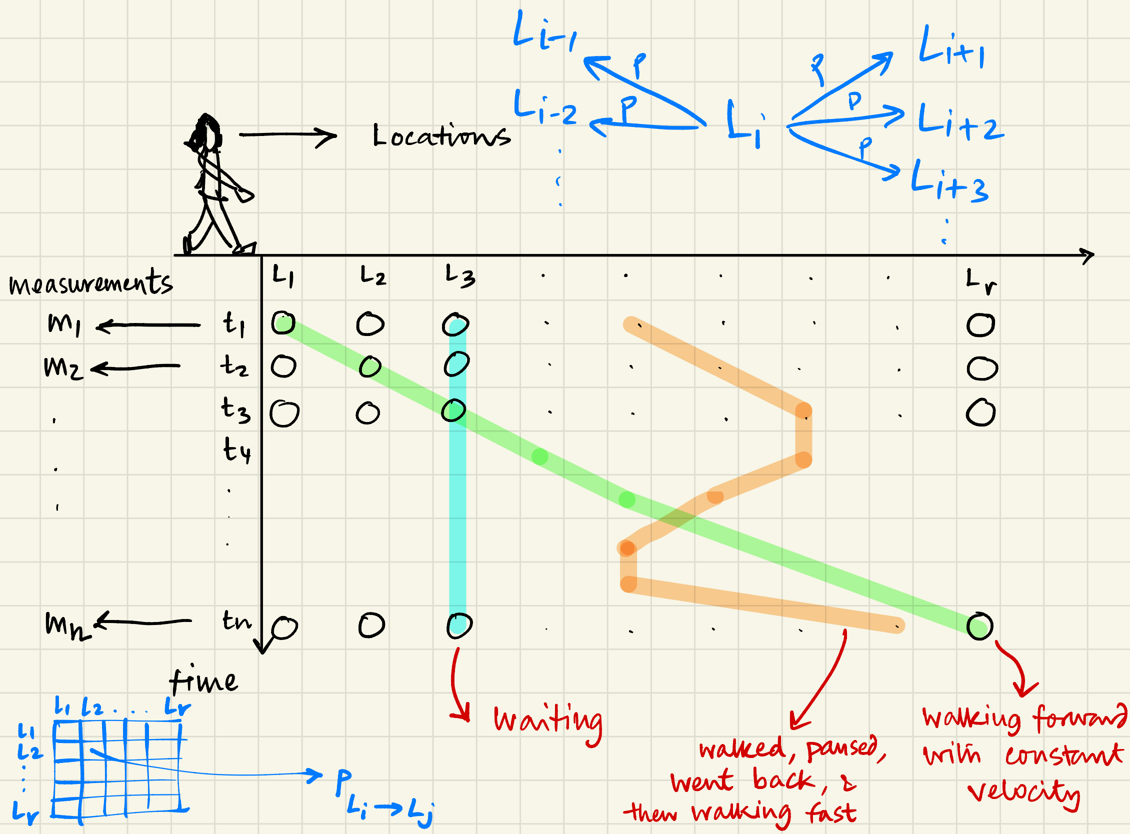
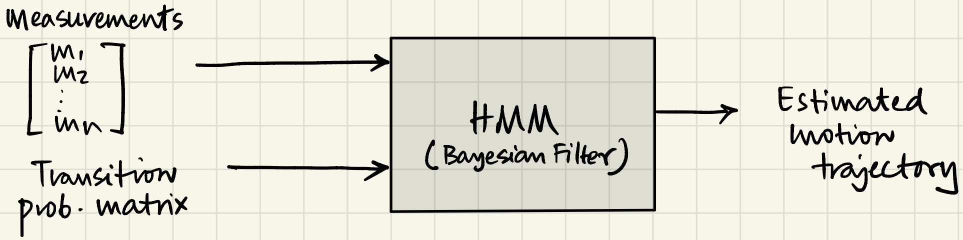
HIDDEN MARKOV MODELS (HMM)





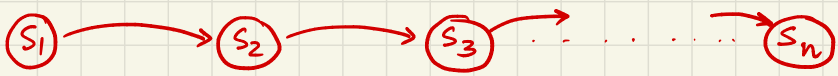


(7) Formulating the State transition diagram :

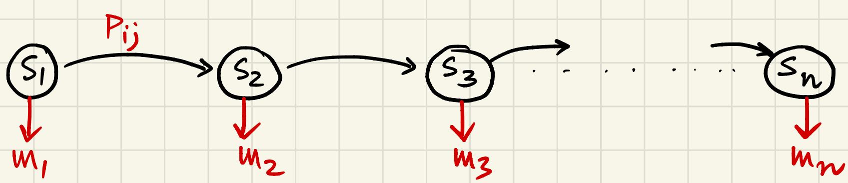
Let S_k denote the state of the subject at time k

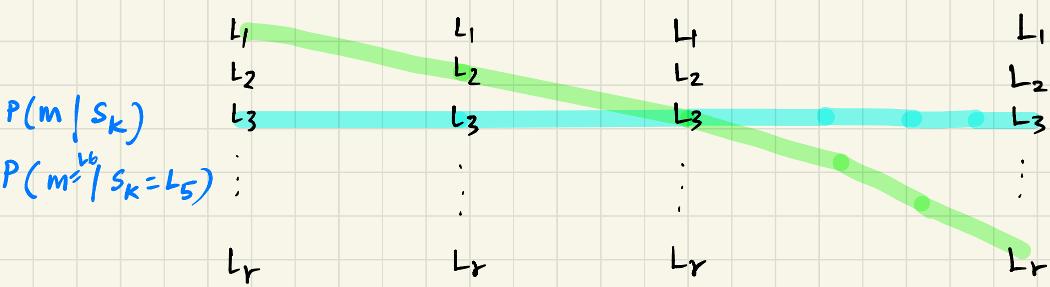
- S_k is a random variable, i.e., $S_k = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix}$ $\forall k \in [1, n]$
- S_k called the "state variable"

The human's walking motion is captured in



And the measurement and motion model is available for each state





(8) Key Question : Where is/was the human at time t_k ?

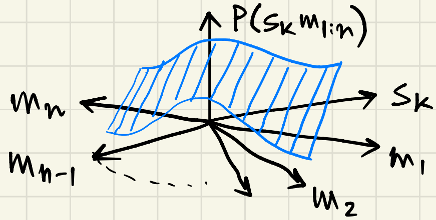
$P(S_k \mid m_{1:k})$ or $P(S_k \mid m_{1:n})$

Is this : Posterior or likelihood ?

Do you have an intuitive feel for $P(s_k \mid m_{1:n})$?
If not, fall back on visualizing them as vectors

$$P\left(s_k = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix} \;\middle|\; m_1 = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix} \quad m_2 = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix} \cdots m_n = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix} \right)$$

$$= \frac{P(s_k, m_{1:n})}{P(m_{1:n})} =$$



$$P\left(m_1 = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix} \quad m_2 = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix} \cdots m_n = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix} \right)$$

→ From this joint distribution (in numerator) you want to know which value of $s_k = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix}$ has the max probability given the $m_{1:n}$ measurements you already have.

→ The denominator is same for all $s_k = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_r \end{bmatrix}$, so only numerator matters. But we don't know that joint distribution.

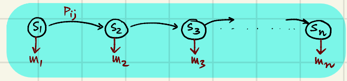
→ Turn this posterior to likelihoods : $\dfrac{P(m_{1:n} \mid s_k)\, P(s_k)}{P(m_{1:n})}$

Likelihood ... and that is not hard because it's the sensor's measurement quality

Who cares! We only want to compare the numerator, so ignore denominator.

Hmmm! This depends on where I was last. So f^n of $P(s_{k-1})$

→ Let's do this mathematically now.



$P(m_3 \mid m_1\, m_2\, s_1\, s_2\, s_3)$

$P(s_3 \mid s_1\, s_2) = P(s_3 \mid s_2)$

→ Let's start with a basic result for $P(\underbrace{s_{1:n} \mid m_{1:n}}_{\text{trajectory given measurements}})$

$$P(s_{1:n} \mid m_{1:n}) = \frac{P(s_{1:n}, m_{1:n})}{P(m_{1:n})} \propto P(s_{1:n}, m_{1:n})$$

$$P(s_{1:n}, m_{1:n}) \overset{\text{chain rule}}{=} P(m_n \mid m_{1:n-1}, s_{1:n})\, P(m_{n-1} \mid m_{1:n-2}, s_{1:n}) \cdots$$

$P(ABC) = P(A|BC)\, P(B|C)\, P(C)$

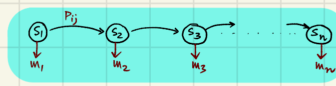
$$\cdots P(m_1 \mid s_{1:n})\, P(s_n \mid s_{1:n-1}) \cdots P(s_2 \mid s_1)\, P(s_1)$$

markov

$$= P(m_n \mid s_n)\, P(m_{n-1} \mid s_{n-1}) \cdots P(m_1 \mid s_1)\, P(s_n \mid s_{n-1}) \cdots P(s_2 \mid s_1)\, P(s_1)$$

$$\boxed{P(s_{1:n}, m_{1:n}) = P(m_1 \mid s_1)\, P(s_1) \prod_{i=2}^{n} P(m_i \mid s_i)\, P(s_i \mid s_{i-1})}$$

→ Now we want $P(s_k \mid m_{1:n})$



$$P(s_k \mid m_{1:n}) \propto P(s_k, m_{1:n}) = P(s_k, m_{1:k}, m_{k+1:n})$$

$$= P(m_{k+1:n} \mid s_k, m_{1:k})\, P(s_k, m_{1:k})$$

Markov

$$= P(m_{k+1:n} \mid s_k)\, P(s_k \mid m_{1:k})\, P(m_{1:k})$$

$$= P(s_k \mid m_{1:k}) \cdot P(m_{k+1:n} \mid s_k)$$

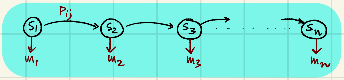
↓ Forward (online)

↓ Probability that murder suspect is at s_k = green st. given $k=4$ recent surveillance camera measurements of main street → wright street → 6th street

↓ Backward (offline)

↓ Probability that 4th to 8th measurement are Neil St. → Kirby road → Lincoln drive → university avenue, given suspect's k^{th} time location s_k = green street.

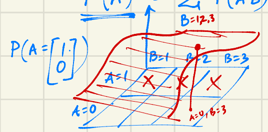
➔ Let's look at the forward component $P(s_k \mid m_{1:k})$



$$P(s_k \mid m_{1:k}) = \frac{P(s_k, m_{1:k})}{P(m_{1:k})} \propto P(s_k, m_{1:k})$$

$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$P(A=1) = \sum_{B=1,2,3} P(A=1, B)$



By marginalizing RHS $= \sum_{s_{k-1}} P(s_k, s_{k-1}, m_{1:k})$

$$= \sum_{s_{k-1}} P(m_k \mid s_k, s_{k-1}, m_{1:k-1})\, P(s_k, s_{k-1}, m_{1:k-1})$$

Markov

$$= \sum_{s_{k-1}} P(m_k \mid s_k)\, P(s_k \mid s_{k-1}, m_{1:k-1})\, P(s_{k-1}, m_{1:k-1})$$

$$P(s_k, m_{1:k}) = \underbrace{\sum_{s_{k-1}} P(m_k \mid s_k)\, P(s_k \mid s_{k-1})\, P(s_{k-1}, m_{1:k-1})}_{\text{call this } \alpha_k}$$

$$\therefore \quad \alpha_k = \sum_{s_{k-1}} P(m_k \mid s_k)\, P(s_k \mid s_{k-1})\, \alpha_{k-1}$$

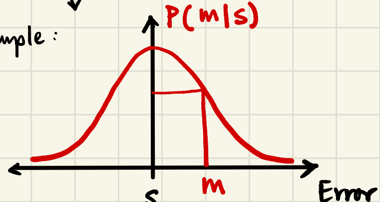
Dynamic program

➔ Initial condition $P(s_1, m_1) = P(m_1 \mid s_1) P(s_1)$ needs to be known.

Errors of sensors derived from their data sheets.

perhaps all locations are equally probable

Example:



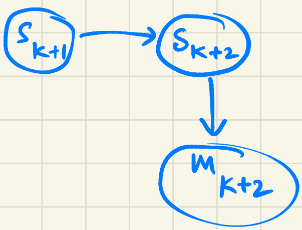
$P(s_k \mid m_{1:k})$

$P(s_k, m_{1:k}) = [0.3, 0.12, \ldots \; r \text{ values}]$

→ Now let's look at the backward part : $P(m_{k+1:n} \mid s_k)$

$$P(m_{k+1:n} \mid s_k) = \frac{P(m_{k+1:n}, s_k)}{P(s_k)}$$

$$= \frac{1}{P(s_k)} \sum_{s_{k+1}} P(m_{k+1:n}, s_k, s_{k+1})$$



$$= \frac{1}{P(s_k)} \sum_{s_{k+1}} P(m_{k+2:n} \mid s_{k+1}, s_k, m_{k+1}) \cdot P(m_{k+1} \mid s_{k+1}, s_k) \cdot P(s_{k+1} \mid s_k) P(s_k)$$

Markov

$$\underbrace{P(m_{k+1:n} \mid s_k)}_{\beta_k} = \sum_{s_{k+1}} \underbrace{P(m_{k+2:n} \mid s_{k+1})}_{\beta_{k+1}} P(m_{k+1} \mid s_{k+1}) P(s_{k+1} \mid s_k)$$

Say LHS $= P(m_{k+1:n} \mid s_k) = \beta_k$

$$\therefore \quad \beta_k = \sum_{s_{k+1}} \beta_{k+1} \, P(m_{k+1} \mid s_{k+1}) \, P(s_{k+1} \mid s_k)$$

Dynamic program again

sensor error distribution

Transition probability matrix

→ How should we initialize this β_k ?

$$\beta_{n-1} = P(m_{n:n} \mid s_{n-1}) = \sum_{s_n} \frac{P(m_n, s_{n-1}, s_n)}{P(s_{n-1})}$$

$$= \frac{1}{\cancel{P(s_{n-1})}} \sum_{s_n} P(m_n \mid s_n, s_{n-1}) P(s_n \mid s_{n-1}) \cancel{P(s_{n-1})}$$

$$\beta_{n-1} = \sum_{s_n} P(m_n \mid s_n) P(s_n \mid s_{n-1})$$ ⇒ Both terms known

$$\beta_{n-2} = \sum_{s_{n-1}} \beta_{n-1} P(m_{n-1} \mid s_{n-1}) P(s_{n-1} \mid s_{n-2})$$

→ Recall original goal : $P(s_k \mid m_{1:n})$ ⇒ offline version

$$P(s_k \mid m_{1:n}) = P(s_k \mid m_{1:k})\, P(m_{k+1:n} \mid s_k)$$

↓ Dynamic prog. ↓ Dynamic prog

Hide

$$\left(\alpha_k = \sum_{s_{k-1}} P(m_k \mid s_k)\, P(s_k \mid s_{k-1})\, \alpha_{k-1} \right)$$

$$\left(\beta_k = \sum_{s_{k+1}} \beta_{k+1}\, P(m_{k+1} \mid s_{k+1})\, P(s_{k+1} \mid s_k) \right)$$

HMM's ⇒ Efficiently identifying the most likely value of a state variable from a large space of computation and possibilities

⇒ Possible to also compute the full trajectory
↳ called Viterbi Decoding