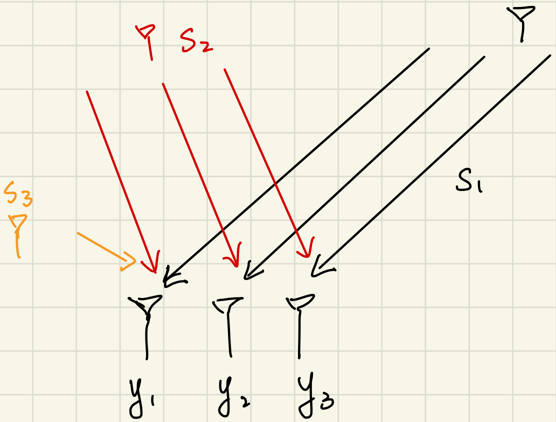
MVDR

Minimum Variance Distortionless Response

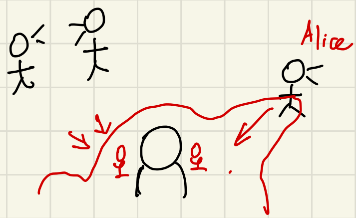
A beamforming algorithm



Problem :
Multiple sources at different angles θ_i

Goal :
Maximize S_1 from θ_1 (known)
Suppress S_2, S_3 at other angles
θ_2, θ_3 are unknown

Analogy:
In the cocktail party, focus on Alice's speech



scale and delay on y_0

$$Y = AS + n = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_d \\ | & & | \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_d \end{bmatrix} + n \ , \quad a_1 = \begin{bmatrix} e^{j0} \\ e^{j\phi_1} \\ \vdots \\ e^{j(N-1)\phi_1} \end{bmatrix}$$

$$w = \begin{bmatrix} w_0 \\ \vdots \\ w_{N-1} \end{bmatrix} \in \mathbb{C}^N$$

Apply response vector w to Y Delay and Sum $w = a_1$

scale and delay on y_{N-1}

$$w^*Y = \underbrace{w^*a_1}_{\text{Keep}} s_1 + \underbrace{\sum_{j=2}^{d} w^*a_j s_j + w^*n}_{\text{minimize}}$$

Formulation

$$\underset{w}{\text{argmin}} \, \|w^*Y\|^2 \quad \text{s.t.} \ \|w^*a_1\| = 1$$

$$[- \ a_i^* \ -] \qquad \uparrow a_i^* \qquad \begin{bmatrix} | \\ a_i \\ | \end{bmatrix}$$

$\|w'a_1\|$
$= \|wa_1\| = 1$

$$a_1 = \begin{bmatrix} e^{j0} \\ e^{j\frac{\pi}{4}} \\ e^{j\frac{\pi}{2}} \\ e^{j\frac{3\pi}{4}} \end{bmatrix} \qquad w' = \frac{1}{4}[e^{-j0} \ e^{-j\frac{\pi}{4}} \ \ldots] \qquad w = [1 \ 0 \ 0 \ 0]$$

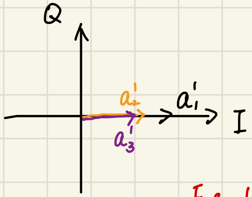
Example of 3 Tx, 4 mics

$$\begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{bmatrix} = \begin{bmatrix} a_1^1 & a_2^1 & a_3^1 \\ \vdots & \vdots & \vdots \\ a_1^4 & a_2^4 & a_3^4 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}^{60^\circ}$$

$s_i \in \mathbb{C}$

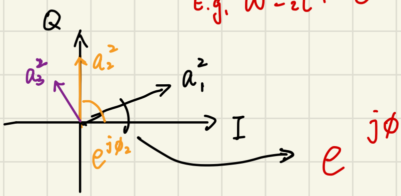
↳ $a_1 = \begin{bmatrix} e^{j0} \\ e^{j\phi_1} \\ \vdots \\ e^{j3\phi_1} \end{bmatrix}$

$y^1 =$

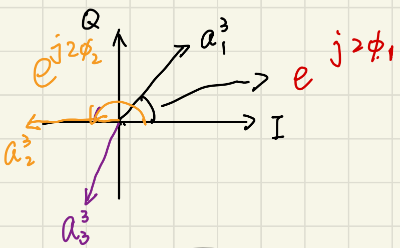


E.g. $w = \frac{1}{2}[1 \;\; e^{-j\phi_1} \;\; 0 \;\; 0]$

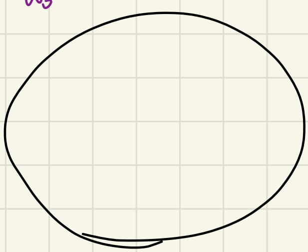
$y^2 =$



$y^3 =$



$y^4 =$



Q : How could we combine signals from 3 antennas s.t. ① s_1 is kept ② s_2 s_3 are suppressed

$$w_1 y_1 + w_2 y_2 + w_3 y_3 = \sum_{n=0}^{N-1} w_n y_n$$

$\sum_{n=0}^{N-1} w_n a_1^n$ is kept

$\sum_{n=0}^{N-1} w_n a_d^n$ are small $\forall d \neq 1$

Formulation :

$$\|w^* Y\|^2 = \begin{bmatrix} - w^* - \end{bmatrix}\begin{bmatrix} | \\ Y \\ | \end{bmatrix} \cdot \begin{bmatrix} - Y^* - \end{bmatrix}\begin{bmatrix} | \\ w \\ | \end{bmatrix}$$

$$\underset{w}{\operatorname{argmin}} \; \| w^* \underbrace{Y Y^*}_{R_{YY}} w \| \quad s.t. \; \| w^* a_1 \| = 1$$

R_{YY} recieve signal covariance

Lagarange Multiplier

$$\text{Solution} \quad w = \frac{R_{YY}^{-1} a_1}{a_1^* R_{YY}^{-1} a_1}$$

make $\| w^* a_1 \| = 1$

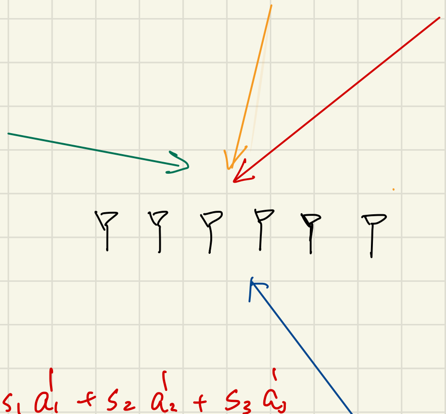
$$w^* a_1 = \frac{a_1^* R_{YY}^{-1}}{a^* R_{YY}^{-1} a_1} \cdot a_1 = 1$$

$$\begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} - a_1^* - \\ - a_2^* - \\ - a_3^* - \end{bmatrix}$$

MUSIC algorithm — MUltiple Signal Classification

Problem : Find θ_k for $k = 4$ Txs w/ 6 mics

An AoA detection algorithm

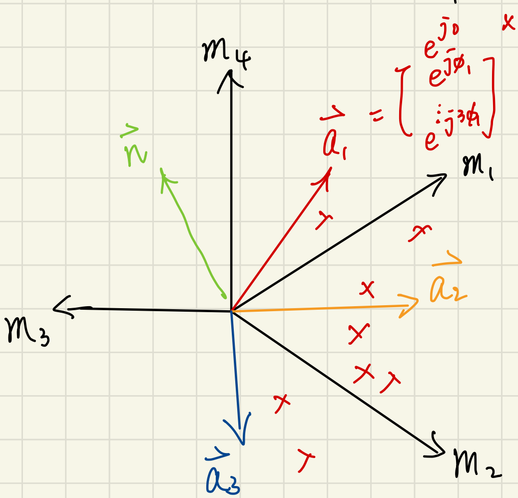


$Y = s_1 a_1 + s_2 a_2 + s_3 a_3$

⇒ Delay and sum does not work well for

Why? $a_i a_j s_j \neq 0$

Microphone received space (complex, but here we draw real)

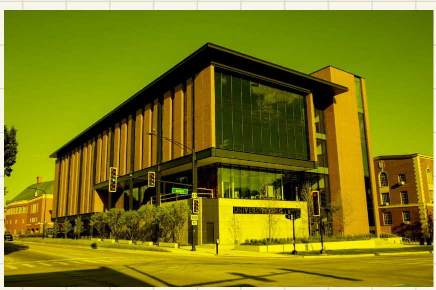


If a_i a_j are correlated

$\sum_j a_i a_j s_j$ is large

Idea : Find the "noise" space

Lack blue color →



span (R,G)

Sub-space based AoA : MUSIC algorithm

$$Y = A\bar{s} + \bar{n}$$

$$YY^H = (As + n)(As + n)^H$$

$$= (As + n)(s^H A^H + n^H)$$

$$= Ass^H A^H + As.n^H + ns^H A^H + nn^H$$

uncorrelated

$$E[YY^H] = E[Ass^H A^H + \cancel{As n^H} + ns^H A^H + nn^H]$$

$N \times N \rightarrow$

$$\underbrace{R_{YY}}_{\text{received signal covariance}} = \underbrace{A R_{ss} A^H}_{\text{signal covariance}} + 0 + 0 + \underbrace{\sigma^2 I}_{\text{noise power}}$$

R_{ss}: $d \times d$

$A = \mathbb{C}^{N \times d}$

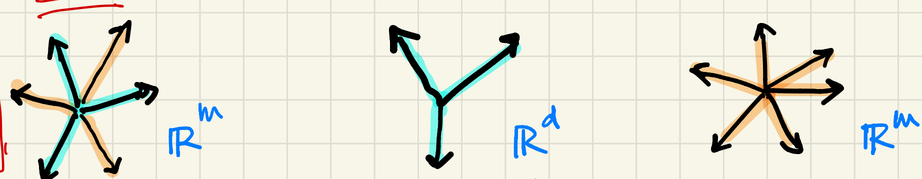
$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_d \\ | & & | \end{bmatrix}$$

Intuition :

$$SVD(R_{YY}) = [U, S, V]$$

$$R_{xx} = A R_{ss} A^H + \sigma^2 I$$

$$a_i = \begin{bmatrix} e^{j0} \\ e^{j\phi_1} \\ \vdots \\ e^{j(N-1)\phi_1} \end{bmatrix}$$



signal + noise space

signal sub-space

$$\begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix} (R_{xx} - \sigma^2 I)\, \bar{e}_{d+1} = 0 = A R_{ss} A^H . \bar{e}_{d+1}$$

$$[s_1\ s_2\ s_3]$$

$\times$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

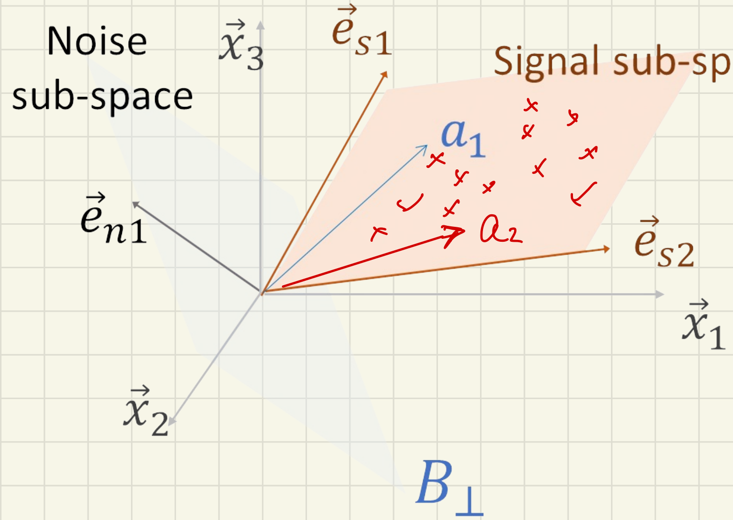
least $(N-d)$ eigenvectors

$\Rightarrow$ noise subspace

Both full rank

$$\therefore A^H . \bar{e}_{d+1} = 0$$

$$A^H = (A^*)^T$$

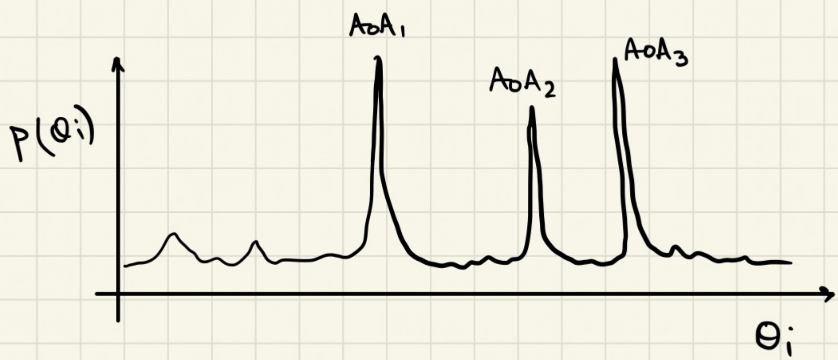


AoA spectrum $P(\theta_i) = \dfrac{1}{\| \underbrace{A^H_{\theta_i}}_{1\times m} \underbrace{e_{noise}}_{m\times(m-d)} \|_{L_1}}$

$$\begin{bmatrix} - A_{\theta_i} - \end{bmatrix} \begin{bmatrix} | & \cdots & | \\ e_{d+1} & & e_m \\ | & & | \end{bmatrix} = \begin{bmatrix} \cdots\cdots \end{bmatrix}$$

good $a_{\theta_i} \perp e_{noise}$

$a_{\theta_i}^* e_{noise} = 0$

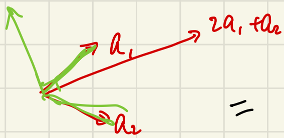


"When you have eliminated the impossible, whatever remains, however impropable, must be the truth" – Sherlock Holmes

Problem : When sources are "correlated"

$$s_2 = \alpha s_1$$

$$Y = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}\begin{bmatrix} s_1 \\ \alpha s_1 \end{bmatrix} + n$$



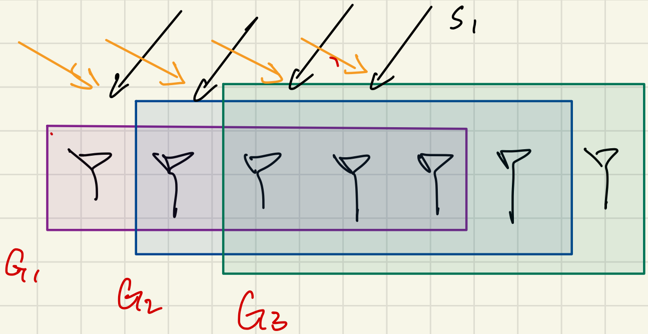
$$= (a_1 + \alpha a_2) s_1 + n$$

1. signal subspace is $< d$ (rank $1 < 2$)
2. $\vec{e}_{d+1} \perp (a_1 + \alpha a_2)$

$P(\theta_2)$ may not be high!

MUSIC spatial smoothing

⇒ Separate the antenna array into groups of the same size (shap)



Note : $Y_1^2 = Y_2^1$

$$A = [a_1^1 \; a_2^1]$$

$$Y_1 = A \begin{bmatrix} s_1 \\ \alpha s_1 \end{bmatrix}$$

$$Y_2 = A \begin{bmatrix} s_1 e^{j\phi_1} \\ \alpha s_1 e^{j\phi_2} \end{bmatrix}$$

$$+)\; Y_3 = A \begin{bmatrix} s_1 e^{j2\phi_1} \\ \alpha s_1 e^{j2\phi_2} \end{bmatrix}$$

$$\sum_{g=1}^{G} Y_g = A \cdot \begin{bmatrix} s_1 \cdot \sum_{g=0}^{G-1} e^{jg\phi_1} \\ \alpha s_1 \cdot \sum_{g=0}^{G-1} e^{jg\phi_2} \end{bmatrix} > \text{uncorrelated}$$

\+ Solve the correlated sources

\- Reduce the # of microphones in each group