Singular Value Decomposition (SVD)

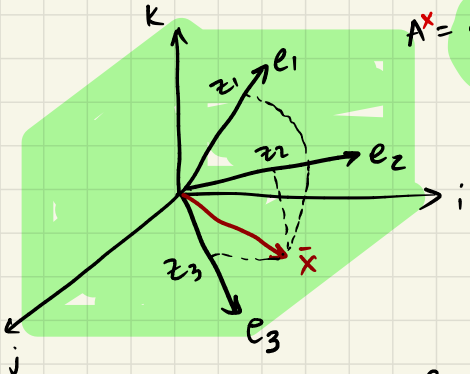
- Recall Eigen-Decomp. $A = S \Lambda S^{-1}$

$$S = \begin{bmatrix} | & | & \\ e_1 & e_2 & \cdots \\ | & | & \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$$

In general,

$e_i \not\perp e_j$

when $A = A^T$, then $e_i \perp e_j$



$$A\mathbf{x} = S \Lambda S^{-1} \mathbf{x}$$

$S^{-1}\mathbf{x}$: project to Eigen vectors

$$S^{-1}\mathbf{x} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\Lambda S^{-1}\mathbf{x} = \begin{bmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \\ \lambda_3 z_3 \end{bmatrix}$$

S: Combine or synthesize back to Identity basis.

① $A_{m\times m}$ and is full rank. $\Longrightarrow S_A$

S is eigen basis of $\mathbb{R}^m$

② $B_{m\times m}$ and is also full rank $\Rightarrow$ spans $\mathbb{R}^m$

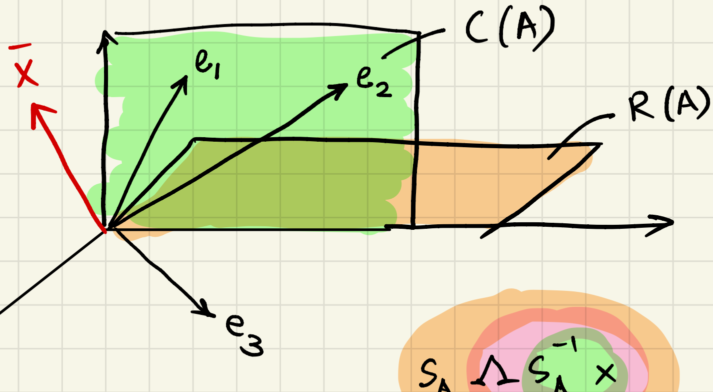
$B_{m\times m} \longrightarrow S_B$

③ say $A_{m\times m}$ is rank deficient.

$$\text{Rank}(A) = r < m$$

$m=3$

$$A = S_A \Lambda_A S_A^{-1}$$



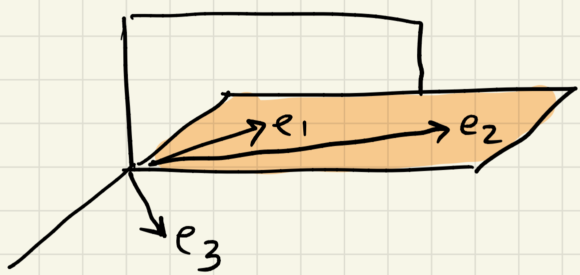
$$S_A \Lambda S_A^{-1} x$$

$\lambda_3 = 0$

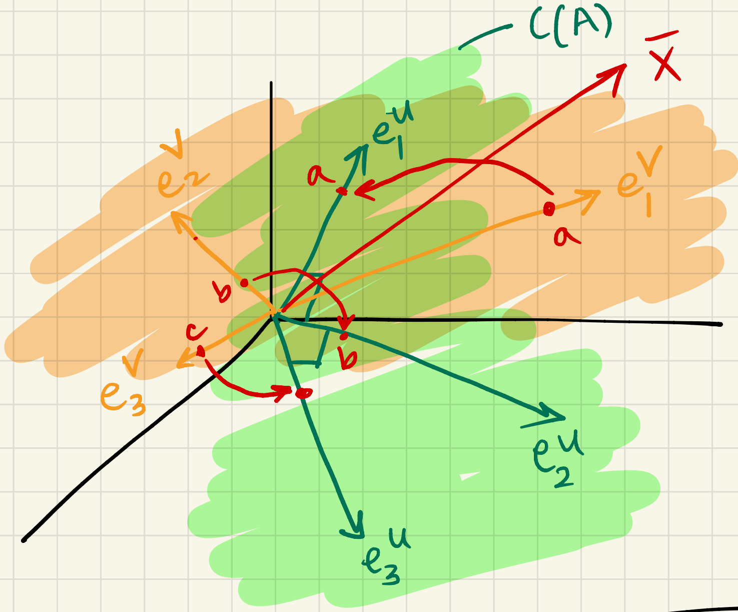
$$Ax = \lambda x$$

$$Ax_n = 0 = 0 \cdot x_n$$

④ $A^T_{m\times m}$ is rank def. $r<m$, $m=3$



$$S_{A^T} \Lambda_{A^T} S_{A^T}^{-1}$$



Eigen decomp

↓

$$A = S \Lambda S^{-1}$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & & 0 \\ 0 & 0 & \ddots & \sigma_m \end{bmatrix}$$

$$A x = U \Sigma V^{-1} x$$

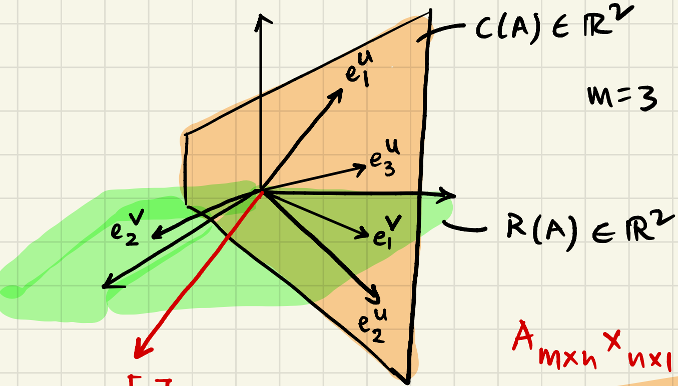
orthogonal eigen basis of C(A)

singular values of A.

orthogonal eigenbasis of R(A)

$$\begin{bmatrix} | & | & | \\ e_1^u & e_2^u & e_3^u \\ | & | & | \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \longrightarrow \text{a vector} = A\bar{x}$$

Consider $A_{m\times n} = \underbrace{\overbrace{\begin{bmatrix} \text{Thin} \\ \text{matrix} \end{bmatrix}}^{n=2}}_{m=3}$ → Full rank



$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$Ax = U \Sigma_1 V^{-1} x$$

$$A_{m\times n} x_{n\times 1} = U_{m\times m} \Sigma_{1\,m\times n} V^T_{n\times n} x_{n\times 1}$$

$$Ax = \begin{bmatrix} | & | & | \\ e_1^u & e_2^u & e_3^u \\ | & | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & e_1^v & - \\ - & e_2^v & - \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a vector in green 2D space which is R(A)

a vector formed by synthesizing e_i^u with the 3D vector from here

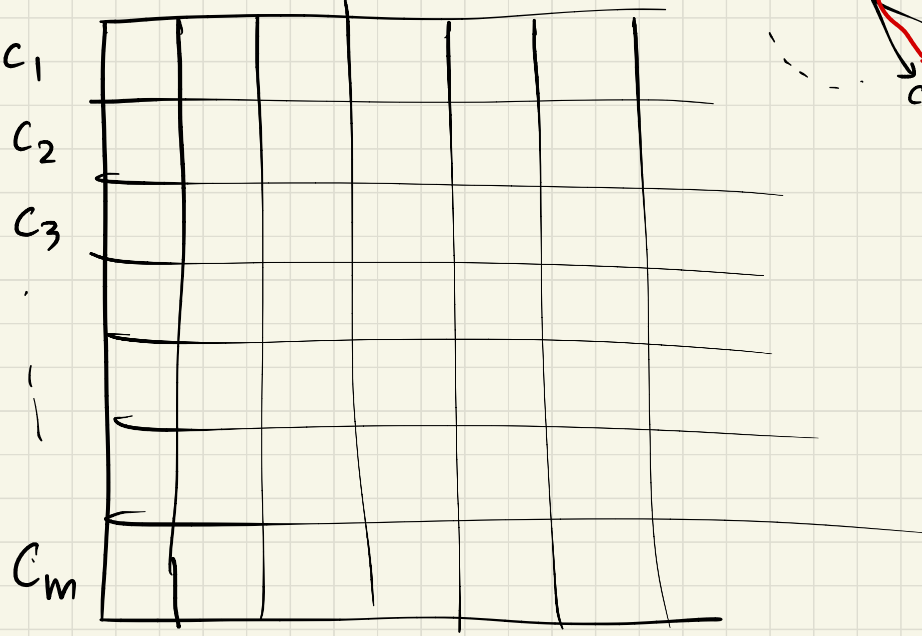
a vector that lives in the green R(A) BUT is a 3D vector

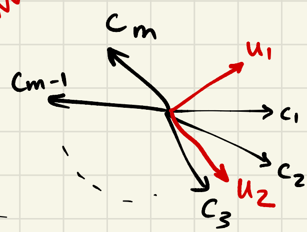
$$\begin{bmatrix} \sigma_1 a \\ \sigma_2 b \\ 0 \end{bmatrix}$$

John

5	3	2	1	5		4?	?	?
Titanic	Matrix	.	–	–	–	–		

Terminator





eigenusers $\Rightarrow e_1 \quad e_2 \quad \ldots \quad e_n$

$$\text{John}\begin{bmatrix} \\ \\ \end{bmatrix} = w_1\begin{bmatrix} e_1 \end{bmatrix} + w_2\begin{bmatrix} e_2 \end{bmatrix} + \cdots w_n\begin{bmatrix} e_n \end{bmatrix}$$

score for terminator → 3 5 1

John's predicted score for terminator

is $w_1 \cdot 3 + w_2 \cdot 5 + \dots w_n \cdot 1$

$\cong 4.2$

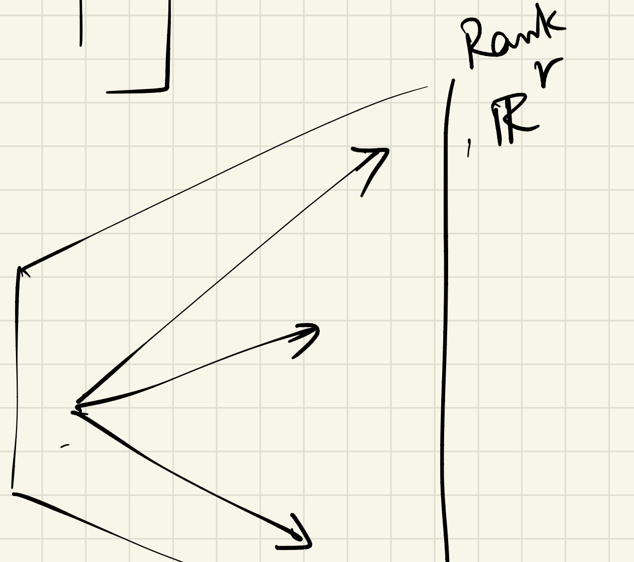
After Class

$$U_{users} = {}_{m}\begin{bmatrix} | & | & & | \\ e_1^u & e_2^u & \cdots & e_m^u \\ | & | & & | \end{bmatrix}^{m}$$

John $= m \times 1$ vector

$$U^{-1}[\text{John}] \equiv U^{-1}_{m\times m} \cdot J_{m\times 1}$$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} | & | & | & & | \\ u_1 & u_2 & u_3 & \cdots & u_n \\ | & | & | & & | \end{bmatrix} \end{array}$$



$e_1 \quad e_2 \quad \cdots \quad e_r \quad e^n_{r+1} \quad e^n_{r+2} \quad \cdots \quad e^n_{r+m}$

AA^T r eigenvectors.

$$\begin{aligned} Ax_1 &= \lambda x_1 \\ Ax_2 &= \lambda x_2 \\ &\vdots \\ Ax_r &= \lambda x_r \end{aligned}$$

$$A_{m \times m} \cdot x_{m \times 1} = \lambda_{1 \times 1} \cdot x_{m \times 1}$$

$$\boxed{A_{m \times n} \cdot x_{n \times 1} = \lambda_{1 \times 1} \cdot x_{m \times 1}}$$

Eigen Decom not possible.

$$A = U \Sigma V^{-1}$$