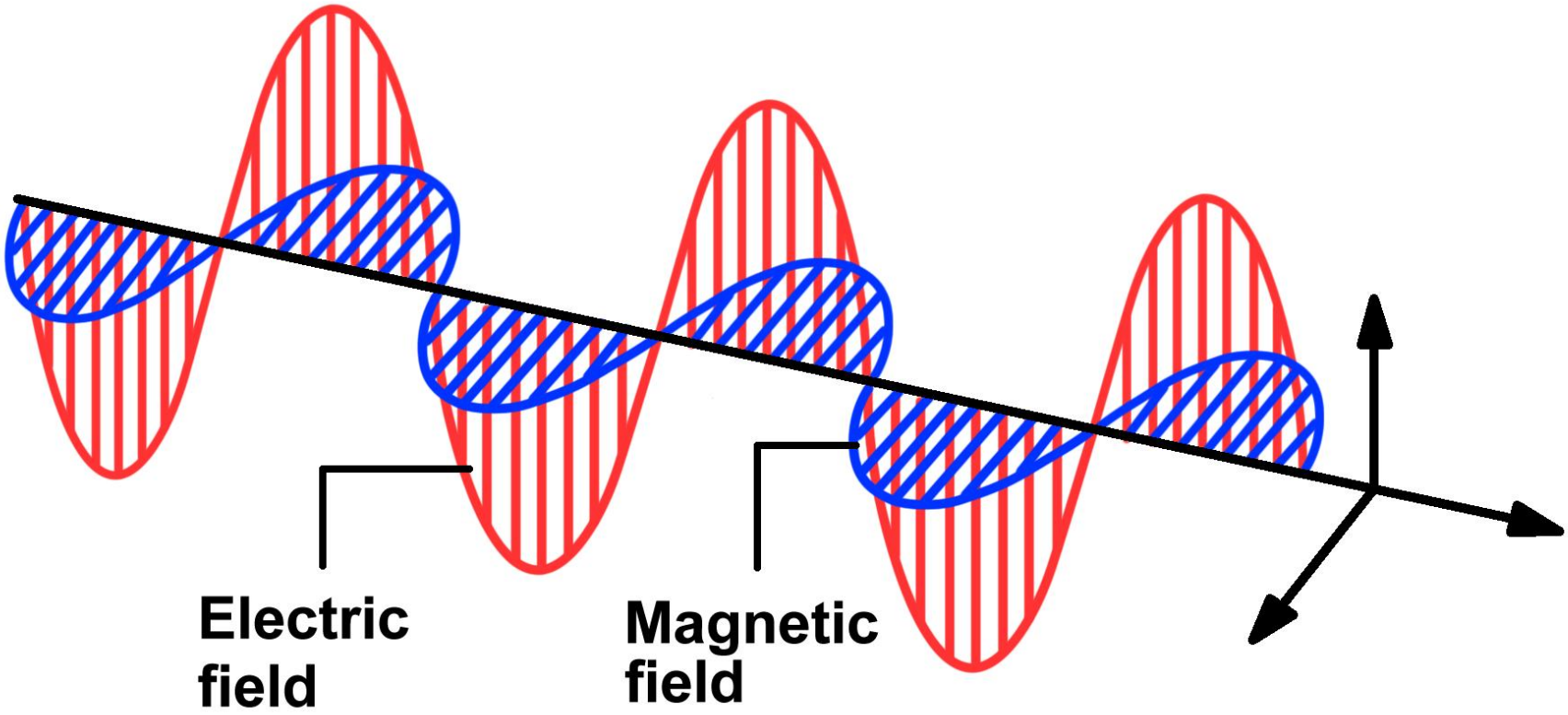


ECE 329: Fields and Waves I

4. Divergence and Curl

What we learned so far

- Electrostatics:
  - Coulomb's Law

$$\vec{F} = q\vec{E}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r}$$

  - Principle of Superposition $\vec{F}_{total} = q\vec{E}_{total} = \sum_n \frac{Q_n}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r_n}|^2}\frac{\vec{r}-\vec{r_n}}{|\vec{r}-\vec{r_n}|}$

- Lorentz Force = force acting on a charge moving with velocity v in the presence of **E** and **B** fields. $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

- Gauss' Law: $\oint_S \vec{D} \cdot d\vec{S} = Q_{encl} = \int_V \rho dV$ or $\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\epsilon_0}$

  - Always holds, independent of surface, how many charges
  - Displacement vector pertains to how an electric field influences the organization of charges in a medium.

$$[\vec{D}] = \frac{C}{m^2}$$

$$[\vec{E}] = \frac{V}{m}$$

$$\vec{D} = \epsilon_0\vec{E}$$

Food for Thought

Imagine that extraordinarily precise new measurements have revealed an error in Coulomb's law and the actual force between any two charges q_1 and q_2 is:

$$F_{12} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1 - \sqrt{\alpha r_{12}}}{r_{12}^2} \mathbf{e_r}$$

*What is the appropriate electric field **E** surrounding a charge q?*

A) $E = \frac{q_1 q_2 q}{4\pi\epsilon_0} \frac{1 - \sqrt{\alpha r_{12}}}{r_{12}^2} \mathbf{e_r}$

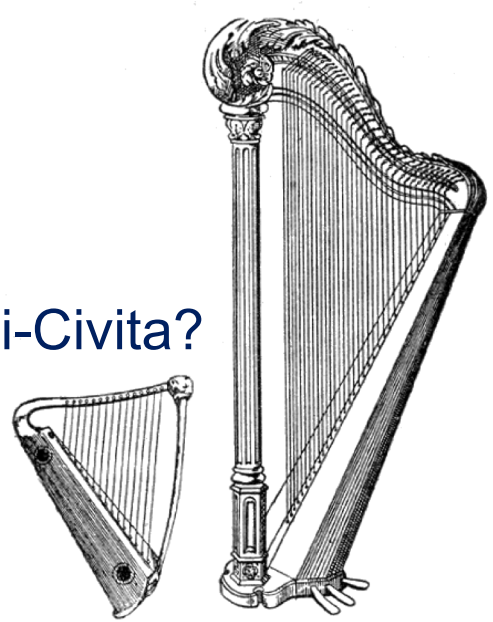
B) $E = \frac{q}{4\pi\epsilon_0} \frac{1 - \sqrt{\alpha r_{12}}}{r_{12}^2} \mathbf{e_r}$

C) $E = \frac{q}{4\pi\epsilon_0} \frac{1 - \sqrt{\alpha r_{12}}}{r_{12}} \mathbf{e_r}$

D) None of the above

∇ (nabla/del) Operator

- **The term Nabla:**
  - Confusion over who coined the term: Maxwell? Levi-Civita?
  - Named after the Egyptian Harp
  - Hebrew: Nevel/nebel



Harps, p. 984.

∇ (nabla/del) Operator

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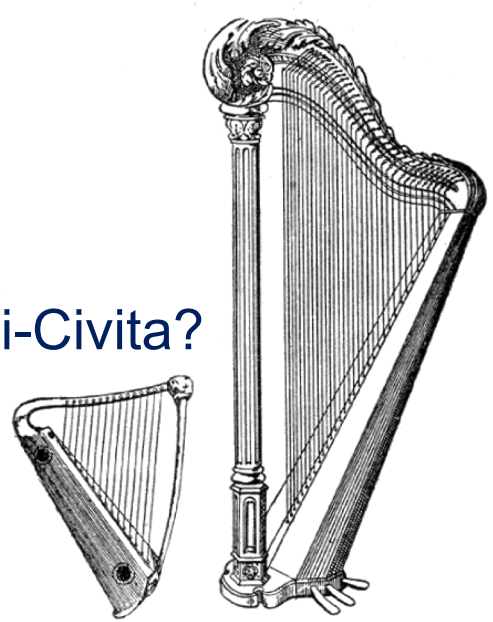
  - Sir William Rowan **Hamilton** was the first to use the Del symbol, or nabla, in a mathematical context in 1837 (Transaction of the Royal Irish Academy).
  - 1846 in Proceedings of the Royal Irish Academy, Hamilton again used the nabla, and this time defined as the del operator that we know today (with one difference).



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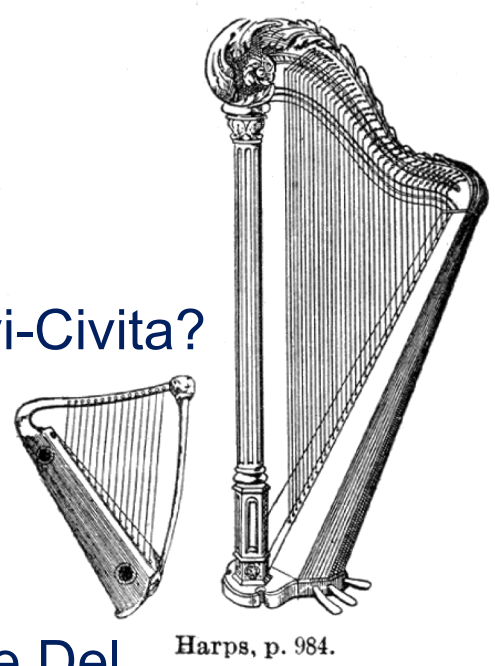
Harps, p. 984.

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$$

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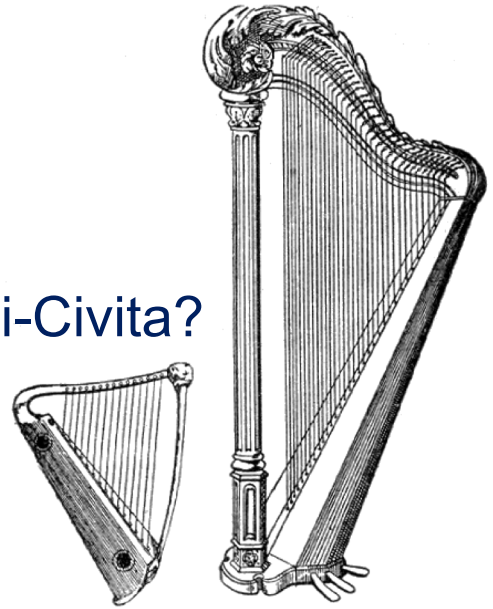
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$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

- **It is an operator!** By itself, it has no real meaning!

Operations with ∇ (nabla/del) Operator

1. Divergence Operator $\nabla \cdot \vec{A}$, $\vec{A}$ = vector field
2. Curl Operator $\nabla \times \vec{A}$, $\vec{A}$ = vector field
3. Gradient Operator ∇f , $\text{grad}(f)$, f = scalar
4. Laplacian Operator $\nabla^2 \vec{A}$, $\nabla^2 f$
5. D'Alembertian Operator $\Box f$

* Returns a scalar field
↳ $\nabla \cdot \vec{E}$ → maps a vector field into a scalar field.

1. Divergence Operator

$$\nabla \cdot \vec{E} = \underbrace{\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}}_{\nabla} \cdot \underbrace{\{E_x, E_y, E_z\}}_{\vec{E}}$$

$$= \underbrace{\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)}_{\nabla} \cdot \underbrace{\left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right)}_{\vec{E}}$$

$$= \underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\text{scalar}}$$

Example: $\vec{E} = (3x, y^2 \sin x) = \hat{x} \cdot 3x + \hat{y}\, y^2 \sin x$

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \cancel{\frac{\partial E_z}{\partial z}}^{\,0} = \underbrace{3 + 2y \sin x}_{\text{scalar}}$$

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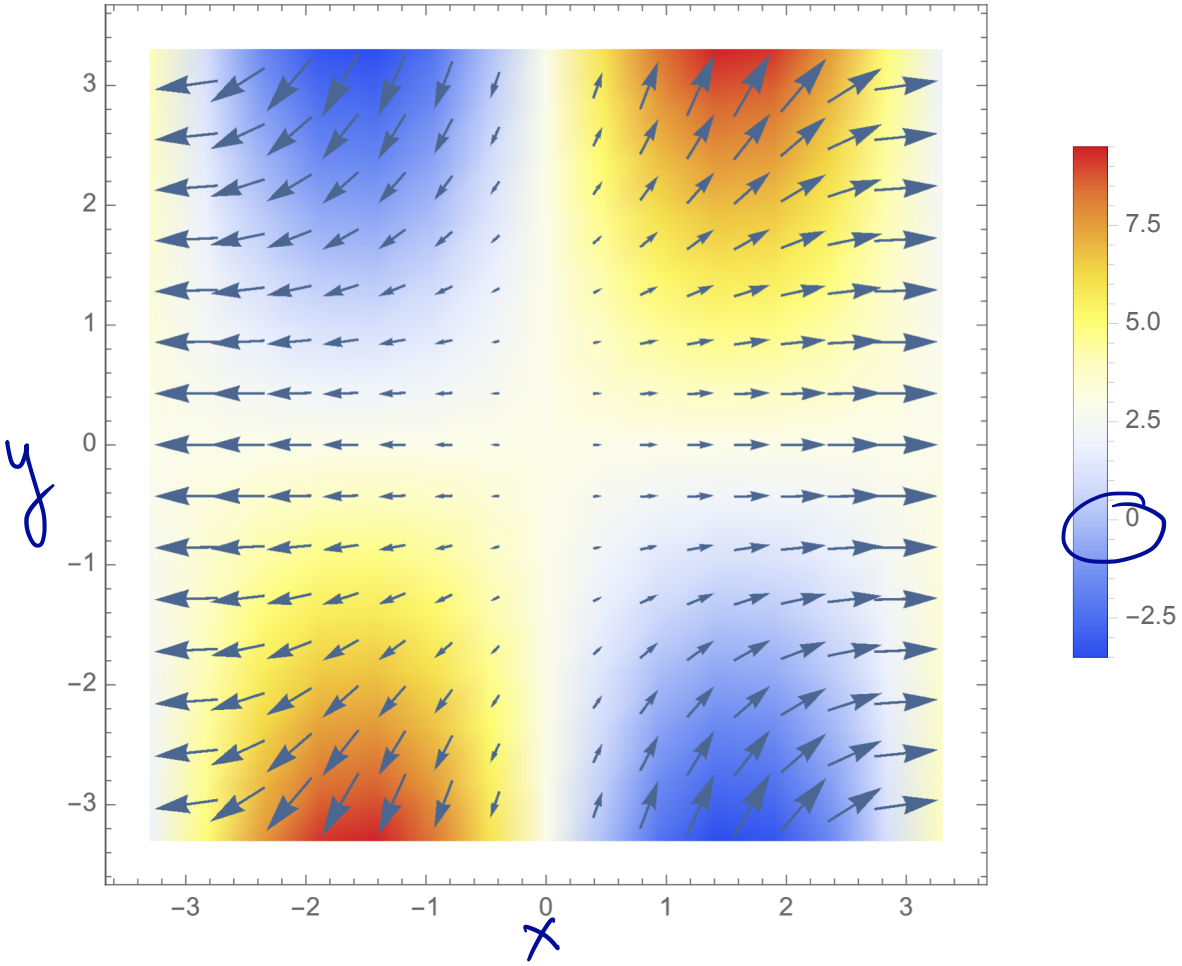
$$\vec{E} = (3x, y^2 \cdot sin(x))$$

$$\nabla \cdot \vec{E} = 3 + 2y \cdot sin(x)$$

1. Divergence Operator



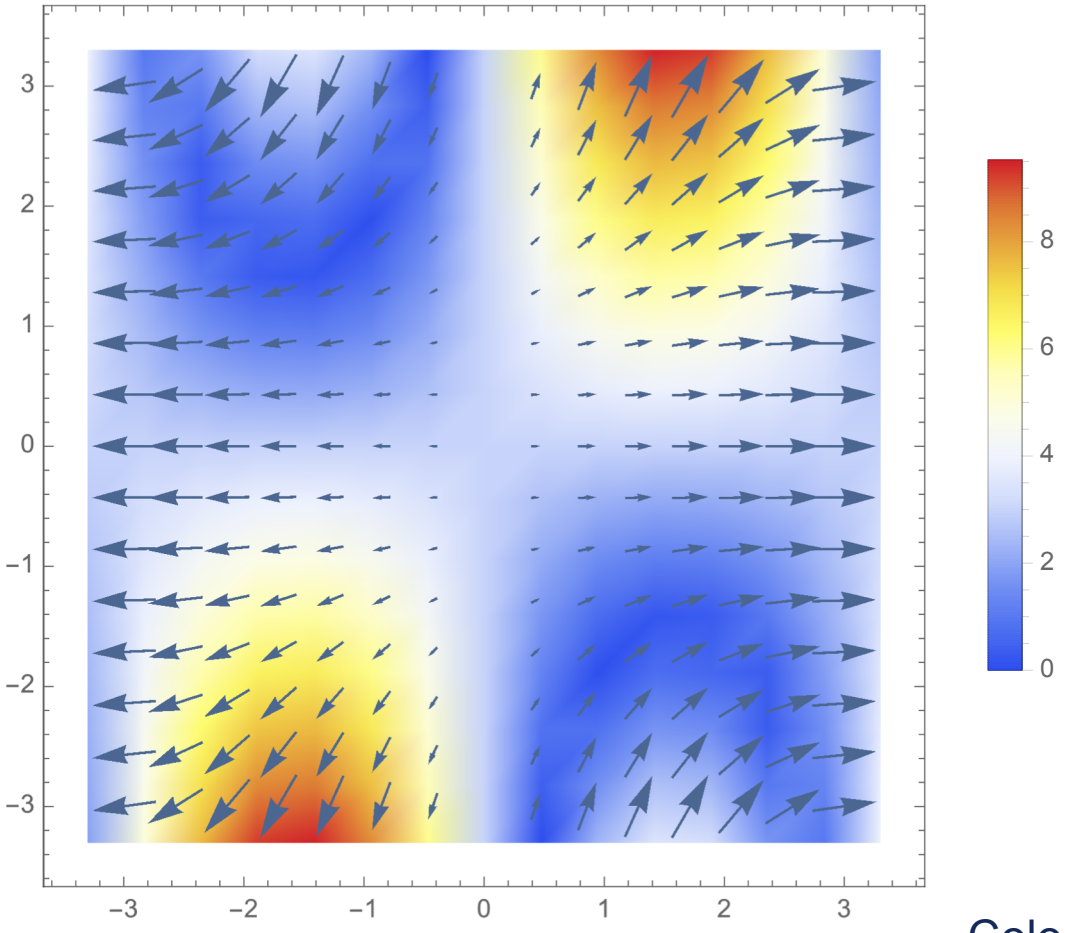
- The divergence gives us the **change of the vector field with position,** in the direction of that position.
- The scale of the divergence is simply how fast the field is changing.

[color=DivE; arrows= E field]

$$\vec{E} = (3x, y^2 \cdot sin(x))$$

$$\nabla \cdot \vec{E} = 3 + 2y \cdot sin(x)$$

1. Divergence Operator



- The bluest regions correspond to points where the divergence is smallest. We can see that these regions can occur where the vector field is changing rapidly, as long as the field magnitude is NOT changing in the direction of the vector.
- In other words, if the vector in field is changing MAGNITUDE but NOT DIRECTION, its divergence will be large.
- If however, the vector is changing direction, but not magnitude, the **divergence will be small.**

Color = absolute value of the divergence.

1. Divergence Operator

Example: Use the result for E for a slab of charge with w=1, ρ=2ϵ_0. Calculate $\nabla \cdot \vec{E}$

$$\vec{E} = \begin{cases} -\hat{z}\frac{\rho w}{2\epsilon_0} & z \le -\frac{w}{2} \\ \hat{z}\frac{\rho z}{\epsilon_0} & -\frac{w}{2} < z < \frac{w}{2} \\ \hat{z}\frac{\rho w}{2\epsilon_0} & z > \frac{w}{2} \end{cases} = \begin{cases} -\hat{z} & z \le -\frac{w}{2} \\ \hat{z}2z & -\frac{w}{2} < z < \frac{w}{2} \\ \hat{z} & z > \frac{w}{2} \end{cases}$$

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \begin{cases} 0, & z \le -\frac{w}{2} \\ 2, & -\frac{w}{2} < z < \frac{w}{2} \\ 0, & z > \frac{w}{2} \end{cases}$$

$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}$: 0

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$\nabla \cdot \vec{E} = \frac{2\rho}{\epsilon_0}$ $\quad \rho = 2\epsilon_0$

* Return a vector, $\nabla \times \vec{A}$ maps a vector into another vector

2. Curl Operator

$$\nabla \times \vec{E} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \times \left\{E_x, E_y, E_z\right\} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right)$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

Example: $\vec{E} = (3x, y^2 \sin x) = (3x, y^2 \sin x, 0)$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x & y^2 \sin x & 0 \end{vmatrix} = \hat{x}(0) - \hat{y}(0) + \hat{z}\, y^2 \cos x$$

$\nabla \times \vec{E} \perp \vec{E}$

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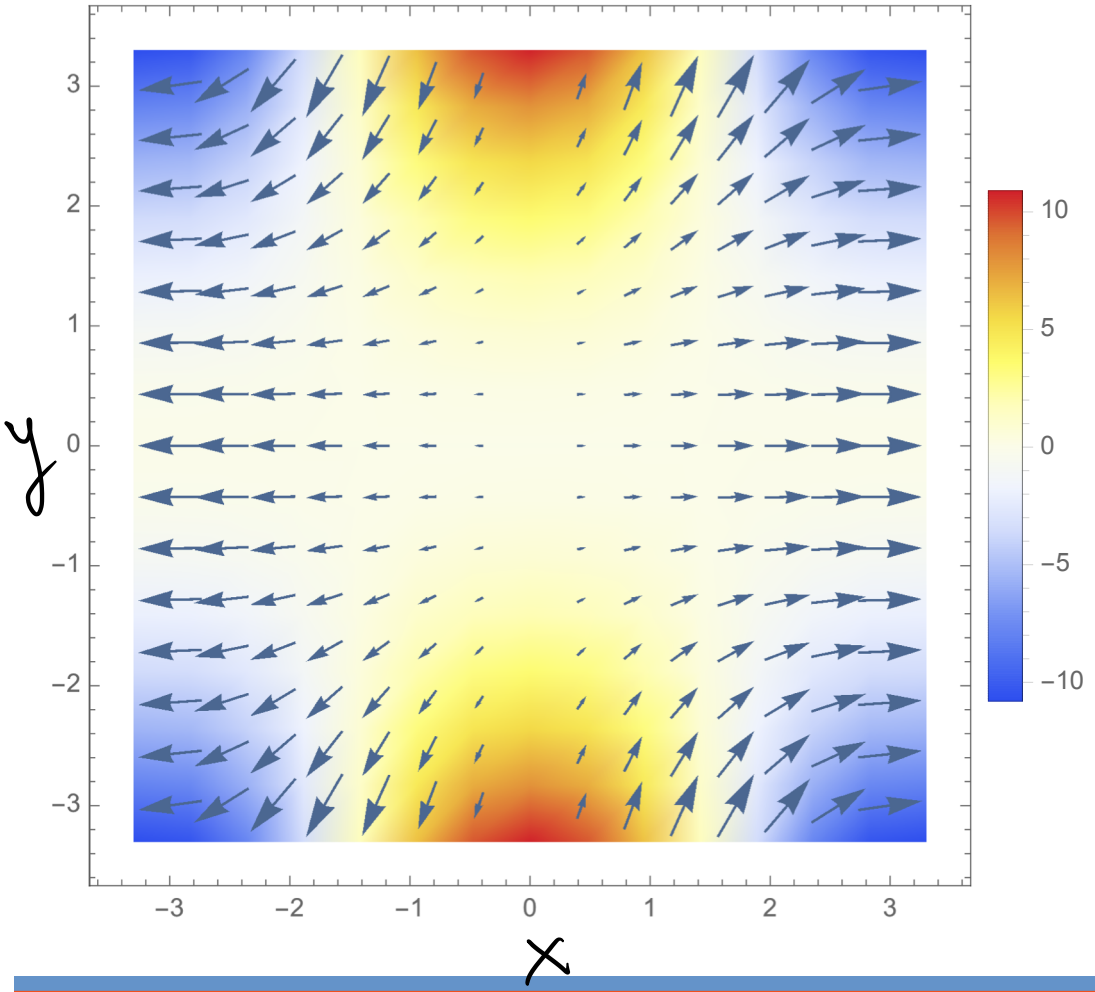
2. Curl Operator

$$\vec{E} = (3x, y^2 \cdot sin(x))$$

$$\nabla \cdot \vec{E} = 3 + 2y \cdot sin(x)$$

$$\nabla \times \vec{E} = \hat{z}y^2 \cdot cos(x)$$



- The bright colors line up with the regions where your vectors are turning (rotating) the fastest.

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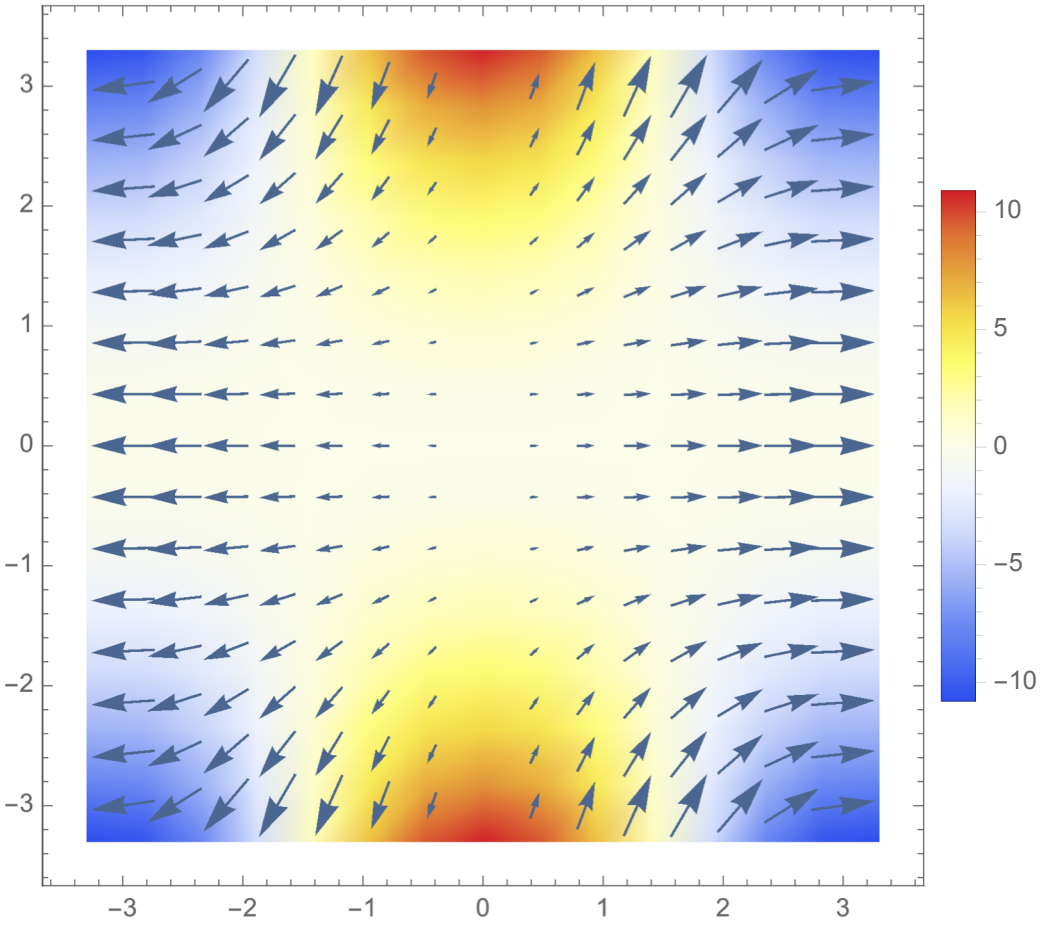

- The bright colors line up with the regions where your vectors are turning (rotating) the fastest.
- Trace the vector field in the upper right hand corner of the plot with the fingers of your right hand (have your fingertips follow the direction of the vectors. **Where is your thumb pointing?**

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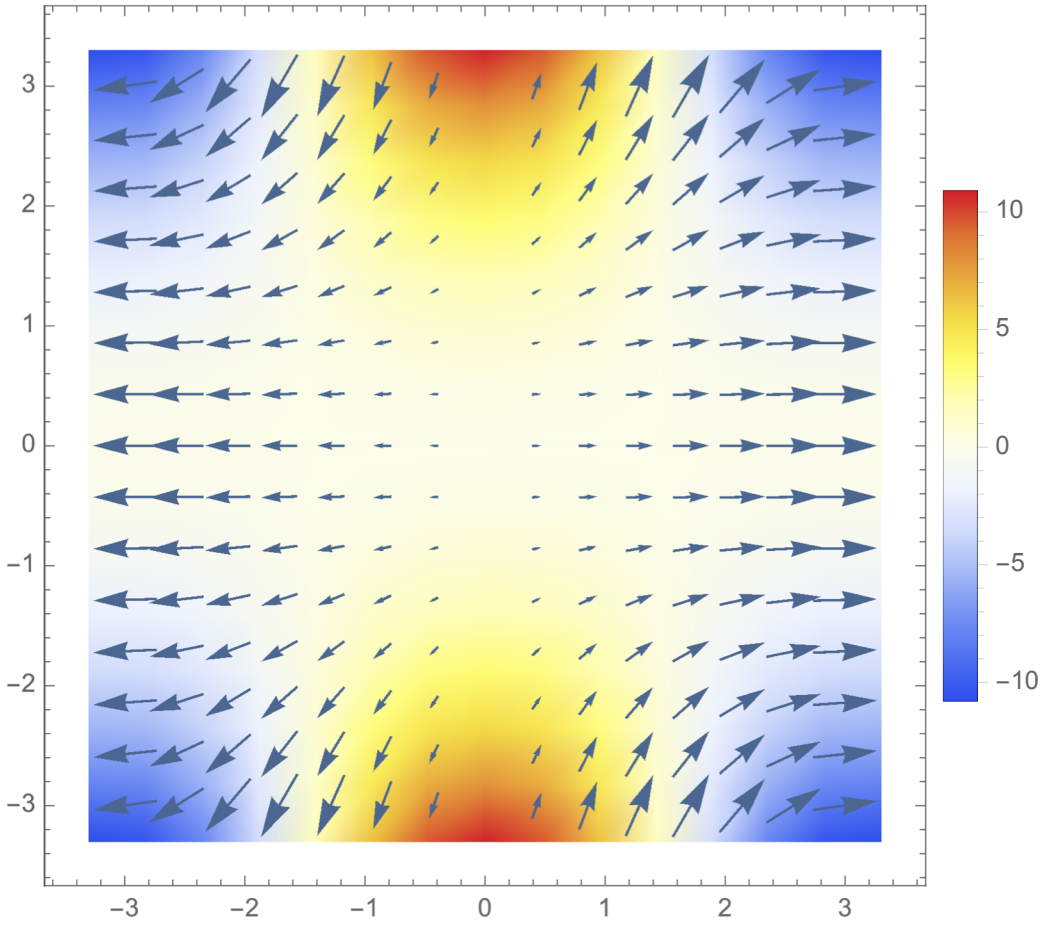

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- Trace the vector field in the upper right hand corner of the plot with the fingers of your right hand (have your fingertips follow the direction of the vectors. **Where is your thumb pointing?**
- **What about at the bottom center of the plot?**

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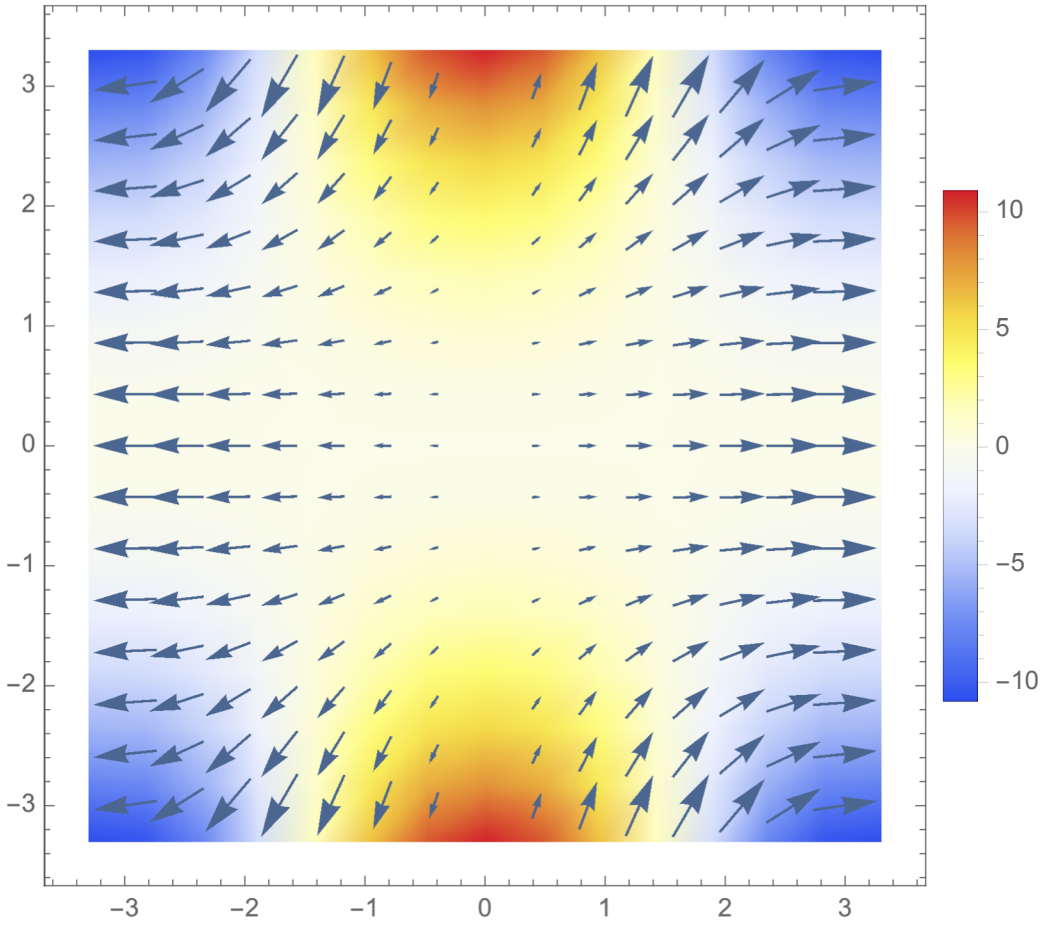

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- Trace the vector field in the upper right hand corner of the plot with the fingers of your right hand (have your fingertips follow the direction of the vectors. **Where is your thumb pointing?**
- **What about at the bottom center of the plot?**
- Note that along the x-axis, your curl is basically zero, even though the vectors are changing size. **Why is that?**

* Maps a scalar field into a vector field.

3. Gradient Operator

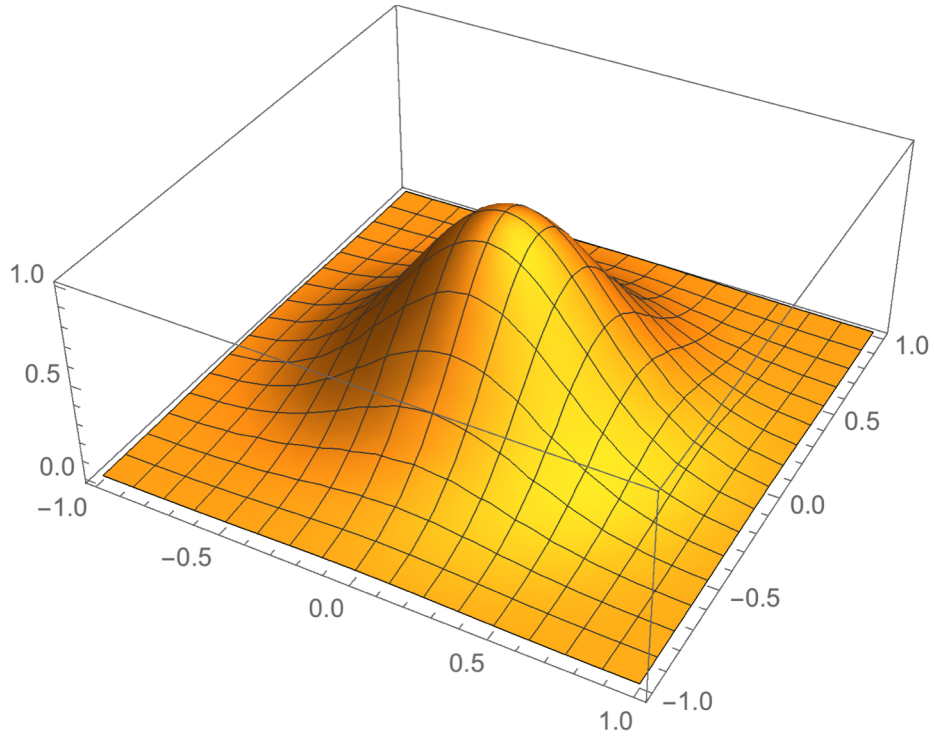
$$\nabla f = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

Example: $f = e^{(-4x^2 - 4y^2)}$

$$\nabla f = \hat{x}\left\{-8x e^{(-4x^2-4y^2)}\right\} + \hat{y}\left\{-8y e^{(-4x^2-4y^2)}\right\}$$

3. Gradient Operator

$$f = exp(-4x^2 - 4y^2)$$



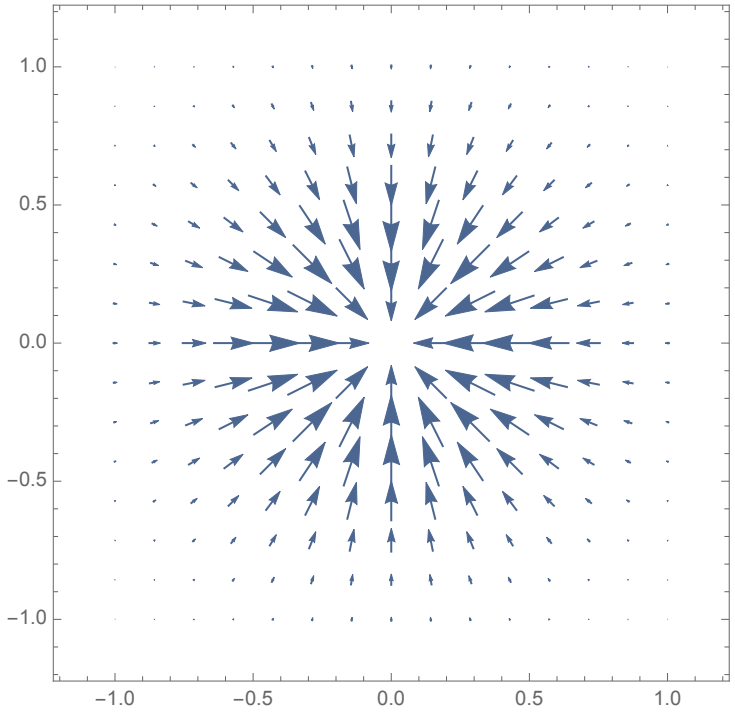
2D Gaussian distribution, but it basically gives us a hill, so we can fairly quickly imagine what the gradient of this function will look like.

3. Gradient Operator

$$f = e^{-4x^2-4y^2}$$

$$\nabla f = \hat{x}(-8xe^{-4x^2-4y^2}) + \hat{y}(-8ye^{-4x^2-4y^2})$$

$$\nabla f = (-8xe^{-4x^2-4y^2}, -8ye^{-4x^2-4y^2})$$



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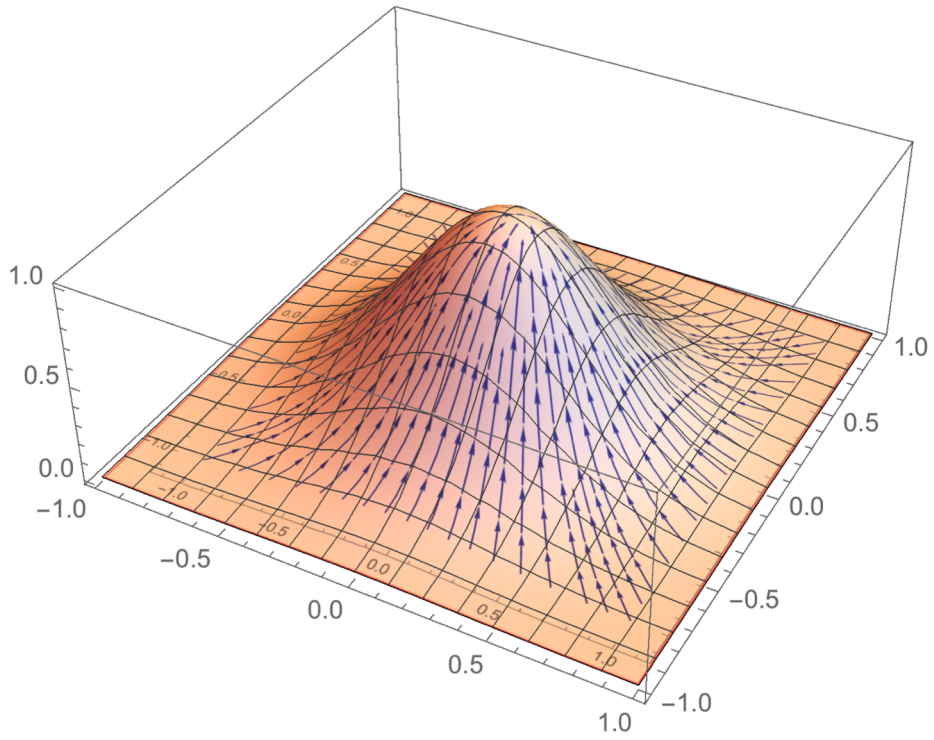


2D Gaussian distribution, but it basically gives us a hill, so we can fairly quickly imagine what the gradient of this function will look like.

Del Operator in Atmospheric Science

Gradient ∇f

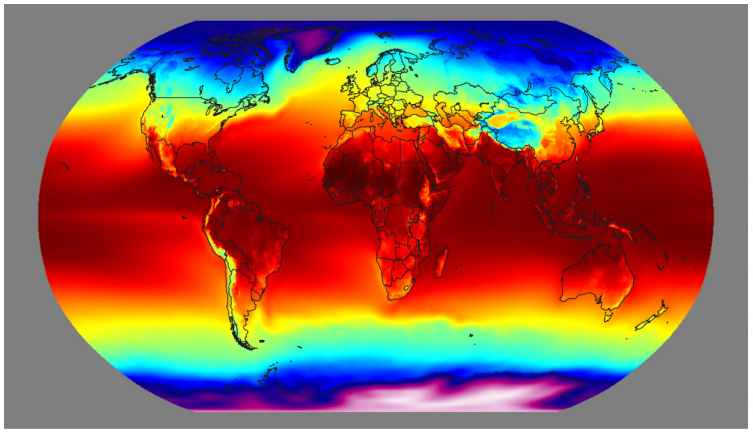
- Solar Insolation
- Land/sea temperature gradients

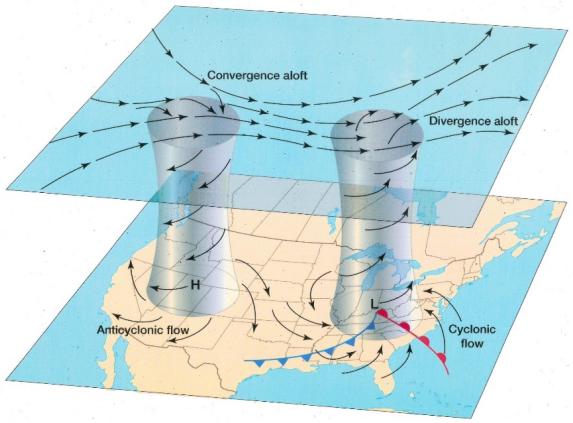
Divergence $\nabla \cdot \vec{v}$

- Synoptic Weather patterns

Curl/Vorticity $\nabla \times \vec{v}$

- Tornado Formation





4. Laplacian Operator

$$\nabla^2 f = \Delta f = \nabla \cdot \underbrace{\nabla f}_{\text{vector}}$$

scalar

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

5. D'Alembertian Operator

aka Wave Operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$