University of Illinois **Spring 2026**

ECE 313: Homework 0: Problems and Solutions

Due: Thursday, January 29 at 02:00:00 p.m.
Reading:

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Thursday, and is due by 2 p.m. on the following Thursday. You must upload handwritten homework to Gradescope. We DO NOT accept typeset homework in LaTeX or Word. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **Usage of generative AI or LLM is strictly prohibited and will be treated as academic integrity violations.**

1. **[MacLaurin, Taylor Series; L'Hopital's Rule]**
   Solve the following problems.

   (a) Prove that $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$ for all $x \neq 1$ and integers $n \geq 1$.
   Solution:

$$1 - x^n = 1 + (x - x + x^2 - x^2 + \cdots + x^{n-1} - x^{n-1}) - x^n \tag{1}$$
$$= (1 + x + x^2 + \cdots + x^{n-1}) - x(1 + x + x^2 + \cdots + x^{n-1}) \tag{2}$$
$$= (1 - x)(1 + x + x^2 + \cdots + x^{n-1}) \tag{3}$$

   Since $x \neq 1$, dividing both sides by $1 - x$ is valid and gives the final answer.

   (b) Assume that n is a positive integer. Show that $1 + x + x^2 + \cdots + x^{n-1} = \lim_{x\to 1} \frac{1-x^n}{1-x}$ when $x = 1$.
   Solution: The left side is clearly n. Using L'Hôpital's rule,

$$\lim_{x\to 1} \frac{1-x^n}{1-x} = \lim_{x\to 1} \frac{-nx^{n-1}}{-1} = n$$

   (c) Assume that $|x| < 1$. Find the sum of the series $1 + x + x^2 + \cdots$ without directly using the Geometric sum formula. **Hint:** think about the limit of the finite sum in part (a) when $n \to \infty$.

Solution:

$$
\begin{aligned}
1 + x + x^2 + \cdots &= \lim_{n\to\infty} 1 + x + x^2 + \cdots x^{n-1} && (4)\\
&= \lim_{n\to\infty} \frac{1-x^n}{1-x} && (5)\\
&= \frac{1}{1-x} && (6)
\end{aligned}
$$

(d) Prove that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

using L'Hôpital's rule.

Solution:

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \frac{\cos(x)}{1} && (7)\\
&= \cos(0) && (8)\\
&= 1 && (9)
\end{aligned}
$$

(e) Prove that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

using the MacLaurin series for $\sin(x)$.

Solution:

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} && (10)\\
&= \lim_{x\to 0} (1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots) && (11)\\
&= 1 && (12)
\end{aligned}
$$

(f) Using the expression you found in part (e), find $d(\frac{\sin(x)}{x})/dx$ at $x = 0$.

Solution: The first derivative of $(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots)$ at $x = 0$ is 0.

2. **[The Binomial Theorem]**
Solve the following problems.

(a) For positive integers n and k, compute the k-th derivative of $(1 + x)^n$ using the chain rule. Use these derivatives to find the Taylor expansion of $(x + 1)^n$ around 0. Then, using what you have found, show that

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Solution: We can find that the k-th derivative is just $n(n-1)(n-2)\cdots(n-k+1)(1+x)^{n-k}$ for $k \leq n$. The last derivative, when $k = n$, is a constant, so our Taylor expansion will have finitely many terms. Constructing this Taylor expansion, we get

$$(1+x)^n = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k$$

Note that the numerator of the coefficient can be written as $\frac{n!}{(n-k)!}$. Making this substitution, we get

$$(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k \quad (13)$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k \quad (14)$$

$$(15)$$

(b) Now consider the function $g(x) = (1-x)^{-n}$. Is the Taylor expansion of this function around 0 the same as part a? Show work explaining why or why not.

Solution: Using the chain rule, the k-th derivative is $n(n+1)(n+2)\cdots(n+k-1)(1-x)^{-n-k}$. Unlike part (a), there is no eventual constant derivative, so our Taylor series will involve infinitely many terms.

Following otherwise similar logic to part (a), we have

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \frac{n(n+1)\cdots(n+k-1)}{k!} x^k$$

(c) Write down the Taylor expansion around 0 for functions $(1-x)^{-1}$ and $(1-x)^{-2}$.

Solution: For $n=1$, $n(n+1)\cdots(n+k-1) = 1\cdot 2\cdots k = k!$. So we have

$$(1-x)^{-1} = 1 + x + x^2 + \cdots$$

For $n=2$, $n(n+1)\cdots(n+k-1) = 2\cdot 3\cdots(k+1) = 1\cdot 2\cdot 3\cdots(k+1) = (k+1)!$. So we have

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots$$

(d) Write down the Taylor expansion around 0 for function $(1+x)^\alpha$, where α is not necessarily an integer.

Solution: Similarly to part (b), the Taylor expansion around 0 has terms of all degrees.

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$$

3. **[Function Extrema]**

Solve the following problems.

(a) Find all maximum and minimum values (if any) of the function $x^{25}(1.00001)^{-x}$ on the interval $(0,\infty)$.

Solution: The function approaches 0 from both ends. Using the first order test, we find that for $25x^{24}(1.00001)^{-x} - x^{25}\ln(1.00001)(1.00001)^{-x} = 0$, it must be the case that the extremum occurs when $x = \frac{25}{\ln(1.00001)}$, or 2500012.5, giving a value of about $1.234 \cdot 10^{150}$. You can see for yourself that this is a maximum, and that, because this is an open interval, there are no other extrema other than the one we just found.

(b) Find all maximum and minimum values (if any) of the function $e^{-|x|}$ on the interval $[-1, 2]$.

Solution: Since the interval is closed, there must be a maximum and a minimum. This function increases as x approaches 0 from both directions, suggesting that the maximum is 1, achieved at $x = 0$. Since the function monotonically decreases with distance from 0, it is clear that the minimum is e^{-2} achieved at $x = -2$.

(c) Find k^* that maximizes the function:

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where $n \geq k$ is a non-negative integer.

Solution: Since k is an integer, we cannot use derivatives to find the maximum. Instead, we can find the value of k which ensures that the ratio:

$$\frac{f(k)}{f(k-1)} = \frac{(n-k+1)p}{k(1-p)} \geq 1 \implies k \leq (n+1)p \implies k^* = \lfloor (n+1)p \rfloor$$

If $(n+1)p$ is an integer then $f(k^*) = f(k^* - 1)$ in which case there are two values of k that maximize the function.

4. **[Definite Integrals]**

Solve the following definite integrals.

(a)

$$\int_{-2}^{1} |x| dx$$

Solution:

$$\int_{-2}^{1} |x| dx = \int_{-2}^{0} -x dx + \int_{0}^{1} x dx \tag{16}$$

$$= -(\frac{1}{2}x^2)|_{-2}^{0} + (\frac{1}{2}x^2)|_{0}^{1} \tag{17}$$

$$= 2 + \frac{1}{2} = \frac{5}{2} \tag{18}$$

(b)

$$\int_{0}^{1} x(1-x^2)^{11} dx$$

Solution: Use u-substitution:

$$u = (1 - x^2) \tag{19}$$

$$du = -2x dx \tag{20}$$

Changing the bounds, we get

$$\int_{0}^{1} x(1-x^2)^{11} dx = \int_{1}^{0} x u^{11} \frac{du}{-2x} \tag{21}$$

$$= \frac{1}{2} \int_{0}^{1} u^{11} du \tag{22}$$

$$= \frac{1}{2} \frac{1}{12} \tag{23}$$

$$= \frac{1}{24} \tag{24}$$

(c)

$$\int_0^1 x^2 e^{-x} dx$$

Solution: Integrate by parts twice.

$$\int_0^1 x^2 e^{-x} dx = (-x^2 e^{-x})|_0^1 + \int_0^1 2xe^{-x} dx \quad (25)$$

$$= -e^{-1} - (2xe^{-x})|_0^1 + \int_0^1 2e^{-x} dx \quad (26)$$

$$= -e^{-1} - 2e^{-1} + (-2e^{-x})|_0^1 \quad (27)$$

$$= -3e^{-1} + (-2e^{-1} + 2) \quad (28)$$

$$= 2 - 5e^{-1} \quad (29)$$

(d)

$$\int_{-10}^{10} x^3 e^{\frac{-x^2}{2}}$$

Solution: The inner function is odd-symmetric, so its integral over symmetric bounds is 0.

5. **[Derivatives and Integrals]**
Let $\frac{d}{dx} f(x) = g(x)$, $f(x) > 0$, $-\infty < x < \infty$, and let C be an arbitrary constant. Which of the following statements are true for all x?

(a)

$$\frac{d}{dx} f(-x) = -g(-x)$$

Solution: True by chain rule.

(b)

$$\frac{d}{dx} f(\frac{x^2}{2}) = xg(\frac{x^2}{2})$$

Solution: True by chain rule.

(c)

$$\frac{d}{dx} e^{f(x^2)} = g(x^2) e^{f(x^2)}$$

Solution: False. $\frac{d}{dx} e^{f(x^2)} = 2xg(x^2)e^{f(x^2)}$.

(d)

$$\int g(-x)dx = f(-x) + C$$

Solution: False. $\int g(-x)dx = -f(-x) + C$.

(e)

$$\int g(\frac{x^2}{2})dx = \frac{f(\frac{x^2}{2})}{x} + C$$

Solution: False. Can be verified by taking the derivative of R.H.S.

(f)

$$\int \frac{g(x)}{f(x)} = \ln(f(x)) + C$$

Solution: True since $f(x)$ is positive. Not generally true.

6. **[Double Integrals]**
Evaluate the following two-dimensional integrals over their specified domains.

(a) Integrate $f(x,y) = \min(x,y)$ over the region $\{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 1\}$
Solution:

$$\int_{y=0}^{1}\int_{x=0}^{2} \min(x,y)dxdy = \int_{y=0}^{1}\int_{x=0}^{y} \min(x,y)dxdy + \int_{y=0}^{1}\int_{x=y}^{2} \min(x,y)dxdy \quad (30)$$

$$= \int_{y=0}^{1}\int_{x=0}^{y} xdxdy + \int_{y=0}^{1}\int_{x=y}^{2} ydxdy \quad (31)$$

$$= \int_{y=0}^{1} \frac{y^2}{2}dy + \int_{y=0}^{1} y(2-y)dy \quad (32)$$

$$= \int_{y=0}^{1} \left(\frac{y^2}{2} + y(2-y)\right) dy \quad (33)$$

$$= \int_{y=0}^{1} \left(\frac{y^2}{2} + 2y - y^2\right) dy \quad (34)$$

$$= (\frac{y^3}{6} + y^2 - \frac{y^3}{3})_0^1 \quad (35)$$

$$= \frac{5}{6} \quad (36)$$

(b) Integrate $f(x,y) = \exp(-\frac{1}{2}(x^2+y^2))$ over the region $\{(x,y) : x^2+y^2 > 4\}$ **Hint:** change to polar coordinates ($x^2+y^2 = 4$ denotes a circle). You may have to do a u-substitution.
Solution:

$$\iint_{x^2+y^2>4} \exp(-\frac{1}{2}(x^2+y^2))dxdy = \int_{r=2}^{\infty}\int_{\theta=0}^{2\pi} r\exp(-r^2/2)d\theta dr \quad (37)$$

$$= 2\pi \int_{r=2}^{\infty} r\exp(-r^2/2)dr \quad (38)$$

Let $u = \exp(-r^2/2)$. Then $du = -rudr$

$$= -2\pi \int_{u=e^{-2}}^{0} ru\frac{du}{ru} \quad (39)$$

$$= -2\pi \int_{u=e^{-2}}^{0} 1du \quad (40)$$

$$= 2\pi \int_{u=0}^{e^{-2}} 1du \quad (41)$$

$$= 2\pi e^{-2} \quad (42)$$